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# 3+1 formulation of light modes in nonlinear electrodynamics

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## ABSTRACT

We present a 3+1 formulation of the light modes in nonlinear electrodynamics described by Plebanski-type Lagrangians, which include post-Maxwellian, Born–Infeld, ModMax, and Heisenberg–Euler–Schwinger QED Lagrangians. In nonlinear electrodynamics, strong electromagnetic fields modify the vacuum such that it acquires optical properties. Such a field-modified vacuum can possess electric permittivity, magnetic permeability, and a magneto-electric response, inducing novel phenomena such as vacuum birefringence. By exploiting the mathematical structures of Plebanski-type Lagrangians, we establish a streamlined procedure and explicit formulas to determine light modes, i.e., refractive indices and polarization vectors for a given propagation direction. We also work out the light modes of the various Lagrangians for an arbitrarily strong magnetic field. The 3+1 formulation advanced in this paper has direct applications to the current vacuum birefringence research: terrestrial experiments using permanent magnets/ultra-intense lasers for the subcritical regime and astrophysical observation of X-rays from highly magnetized neutron stars for the near-critical and supercritical regimes.

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## I. INTRODUCTION

Light propagation in curved spacetimes has long been a topic of great theoretical interest.<sup>1</sup> Recently, the Event Horizon Telescope (EHT) observed the shadow of supermassive black holes from light propagating around the event horizon and has provided a way to understand black hole geometries.<sup>2–4</sup> Nonlinear electrodynamics has also provided a nontrivial background for light propagation. Born and Infeld introduced a nonlinear Lagrangian for the Maxwell scalar and its pseudoscalar, which reduces to the Maxwell Lagrangian in the weak-field limit but leads to significant modifications of the physics of light–matter and light–light interactions in the strong-field limit.<sup>5</sup> A detailed study has been made of the causal properties of the Born–Infeld Lagrangian.<sup>6</sup> Heisenberg and Euler obtained an exact one-loop effective Lagrangian for electrons in a constant electromagnetic field.<sup>7</sup> Later, Schwinger introduced the proper-time method for quantum field theory and obtained a

one-loop effective Lagrangian for spinless charged bosons and spin-1/2 fermions in a constant electromagnetic field.<sup>8</sup>

The prominent features of the Heisenberg–Euler–Schwinger (HES) Lagrangian in quantum electrodynamics (QED) are vacuum polarization<sup>9,10</sup> and Sauter–Schwinger pair production of electrons and positrons.<sup>8,11</sup> Interestingly, in strong electric fields, the vacuum becomes unstable due to pair production, which is a consequence of the imaginary part of the HES Lagrangian.<sup>12,13</sup> The polarized vacuum in a strong electromagnetic field provides a nontrivial background for propagation of probe photons (i.e., light). In recent years, nonlinear electrodynamics (NED) has been intensively studied, partly because of the development of ultra-intense lasers using chirped pulse amplification (CPA) technology<sup>14,15</sup> and partly because of astrophysical observations of highly magnetized neutron stars and magnetars with magnetic fields comparable to or stronger than the critical field strength  $B_c = m^2 c^2 / e \hbar = 4.4 \times 10^{13}$  G.<sup>16</sup> Such a strong electromagnetic field makes the vacuum into a polarized

medium, and this polarized vacuum behaves similarly to a dielectric or ferromagnetic medium. NED in general exhibits, in addition to electric permittivity and magnetic permeability, the magneto-electric effect, in which a magnetic field induces electric polarization while an electric field induces magnetization.<sup>17–19</sup> A strong electric field, on the other hand, creates pairs of charged particles and antiparticles through Sauter–Schwinger pair production.

NED thus exhibits a rich structure of vacuum polarization, involving phenomena such as vacuum birefringence, photon propagation, and the magneto-electric effect. In a strong magnetic field, the polarized vacuum acquires nontrivial refractive indices, which cause birefringence in the photon propagation either along the magnetic field or in the perpendicular plane.<sup>20,21</sup> Light propagation has been intensively studied in the NED of Plebanski-type Lagrangians,<sup>22</sup> according to which a low-energy probe photon satisfies a wave equation in a nonlinear background. Most studies in the literature have employed the covariant four-vector formulation of light propagation (see, e.g., Refs. 20 and 23–27), while the 3+1 formulation has been used for post-Maxwellian theory in weak electromagnetic fields.<sup>28</sup>

In a previous study, we investigated light propagation in a supercritical magnetic field and a weak electric field, in particular for the case of a nonzero electric field along the magnetic field (an “electromagnetic wrench”).<sup>29,30</sup> In contrast to the pure magnetic field case, the electromagnetic wrench introduces magneto-electric effects and thus significantly modifies the light modes. In our previous work,<sup>29,30</sup> we focused on a configuration with parallel electric and magnetic fields to study the effect in the simplest setting. However, given the complicated field configurations around neutron stars or in the focal regions of ultra-intense lasers, we need to extend the previous formulation to an arbitrary field configuration. Furthermore, it is worthwhile making the formulation applicable to other Lagrangians considered in the context of general NED.

In this paper, we introduce the 3+1 formulation of light modes in the NED of Plebanski-type Lagrangians  $\mathcal{L}(F, G)$ ,<sup>22</sup> where  $F$  and  $G$  are the Maxwell scalar and pseudoscalar, respectively:

$$\begin{aligned} F &\equiv \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2), \\ G &\equiv \frac{1}{4} F^{\mu\nu} F_{\mu\nu}^* = -\mathbf{E} \cdot \mathbf{B}. \end{aligned} \quad (1)$$

Here,  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  is the field-strength tensor and  $F^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$  ( $\epsilon^{0123} = 1$ ) is the dual field-strength tensor.<sup>31</sup> We also present a streamlined procedure for determining light modes with explicit formulas, which should be valuable for laboratory experiments and astrophysical observations of vacuum birefringence. As concrete examples, we apply the formulation to the frequently used NED Lagrangians for an arbitrarily strong magnetic field to obtain expressions for the light modes. These expressions can be used as optical tools to test theoretical black holes with NED Lagrangians. We use Lorentz–Heaviside units with  $c = \hbar = 1$ , and thus  $\alpha_e = e^2/4\pi$ , where  $e$  is the elementary charge.

In Minkowski spacetime with metric  $(+, -, -, -)$ , the timelike Killing vector  $\partial_t$  makes the spacetime manifold foliate into one-parameter spacelike hypersurfaces. On each hypersurface  $\Sigma_t$ , the energy–momentum of a photon is given by  $k^\mu = (\omega, \mathbf{k})$  and the field by  $F^{\mu\nu}$ . In curved spacetime, the 3+1 formulation of Maxwell

theory<sup>32–34</sup> has often been preferred for modeling in astrophysics and cosmology. In the 3+1 formulation, the propagation and polarization of probe photons (light) are expressed in terms of directly measurable electromagnetic fields  $(\mathbf{E}, \mathbf{B})$  and the photon direction  $\mathbf{k}$  on each  $\Sigma_t$ . In practice, the 3+1 formulation has the advantage of providing the light modes in general NED in the familiar language of conventional optics, facilitating the design of experiments and observations.

The remainder of this paper is organized as follows. In Sec. II, we present the general 3+1 framework for analyzing light modes in a medium in the presence of strong background fields. Then, in Sec. III, this is compared with the formulation used in materials science to deal with the magneto-electric effect. In Sec. IV, we apply the framework to the NED of Plebanski-type Lagrangians and compare it with the alternative covariant formulation. In Sec. V, we present specific examples with the representative NED Lagrangians: the post-Maxwellian (PM), Born–Infeld (BI), ModMax (modified Maxwell, MM), and the HES QED Lagrangians. For these Lagrangians, the expressions for the light modes in an arbitrarily strong magnetic field are worked out. Finally, in Sec. VI, summarizing the results, we draw our conclusions, discussing the implications of the 3+1 formulation for research on vacuum birefringence, in the context, for example, of terrestrial experiments using ultra-intense lasers in the weak-field regime and of astrophysical observations of highly magnetized neutron stars in the strong-field regime.

## II. 3+1 FRAMEWORK FOR ANALYZING LIGHT PROPAGATION MODES IN A MEDIUM IN THE PRESENCE OF STRONG BACKGROUND FIELDS

In this section, we set up a 3+1 framework to obtain the light propagation modes in an arbitrary medium in the presence of strong background fields. When applied to NED Plebanski-type Lagrangians, as shown in Sec. IV, this framework yields refractive indices and polarization vectors for a given propagation direction.

In classical electrodynamics, the electromagnetic properties of a medium are described in terms of the polarization  $\mathcal{P}$ , the magnetization  $\mathcal{M}$ , the electric displacement  $\mathbf{D}$ , and  $\mathbf{H}$ :<sup>35</sup>

$$\mathbf{D} = \mathbf{E} + \mathcal{P}, \quad \mathbf{H} = \mathbf{B} - \mathcal{M}. \quad (2)$$

To find the modes of weak low-frequency<sup>36</sup> probe light  $(\delta\mathbf{E}, \delta\mathbf{B})$  in the medium polarized and magnetized by a strong slowly-varying background electromagnetic field  $(\mathbf{E}_0, \mathbf{B}_0)$ , we decompose  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\mathbf{D}$ ,  $\mathbf{H}$ ,  $\mathcal{P}$ , and  $\mathcal{M}$  as

$$\mathbf{A} = \mathbf{A}_0 + \delta\mathbf{A}, \quad (3)$$

where  $\mathbf{A}_0$  and  $\delta\mathbf{A}$  denote the background and probe parts, respectively.<sup>21,37</sup> The background part  $\mathbf{A}_0$  is assumed uniform (locally constant) over the characteristic scales of the probe part  $\delta\mathbf{A}$ , which allows us to argue that the relations (2) should be satisfied separately for the background and probe parts:

$$\begin{aligned} \mathbf{D}_0 &= \mathbf{E}_0 + \mathcal{P}_0, & \mathbf{H}_0 &= \mathbf{B}_0 - \mathcal{M}_0, \\ \delta\mathbf{D} &= \delta\mathbf{E} + \delta\mathcal{P}, & \delta\mathbf{H} &= \delta\mathbf{B} - \delta\mathcal{M}. \end{aligned} \quad (4)$$

Because the probe light is weak, only the linear response to the probe light is considered:

$$\delta \mathbf{D} = \epsilon_E \cdot \delta \mathbf{E} + \epsilon_B \cdot \delta \mathbf{B}, \quad \delta \mathbf{H} = \tilde{\mu}_B \cdot \delta \mathbf{B} + \tilde{\mu}_E \cdot \delta \mathbf{E}, \quad (5)$$

where  $\epsilon_E$ ,  $\epsilon_B$ ,  $\tilde{\mu}_B$ , and  $\tilde{\mu}_E$  (rank-2 tensors in the 3D space) are the permittivity ( $\epsilon_E$ ), permeability ( $\tilde{\mu}_B$ ), and magneto-electric response ( $\epsilon_B$  and  $\tilde{\mu}_E$ ) tensors. These tensors, evaluated under condition  $(\mathbf{E}, \mathbf{B}) = (\mathbf{E}_0, \mathbf{B}_0)$ , represent the influence of the strong background fields on the probe light. The tensors  $\epsilon_B$  and  $\tilde{\mu}_E$ , absent in the theory of usual optical materials,<sup>38</sup> represent the magneto-electric effect: the magnetic field induces polarization, and the electric field induces magnetization.<sup>39</sup>

When there are no real charge and current, the Maxwell equations for the probe light become

$$-\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times \delta \mathbf{E}, \quad \frac{\partial \delta \mathbf{D}}{\partial t} = \nabla \times \delta \mathbf{H}, \quad (6)$$

which reduce to

$$\delta \mathbf{B} = \mathbf{n} \times \delta \mathbf{E}, \quad \delta \mathbf{D} = -\mathbf{n} \times \delta \mathbf{H} \quad (7)$$

for a Fourier mode characterized by an angular frequency  $\omega$  and a propagation vector  $\mathbf{k} = \omega \mathbf{n} = \omega n \hat{\mathbf{n}}$ , where  $n$  is the refractive index of the mode, and  $\delta \mathbf{B}$ ,  $\delta \mathbf{E}$ ,  $\delta \mathbf{D}$ , and  $\delta \mathbf{H}$  vary as  $\exp[i\omega(\mathbf{n} \cdot \mathbf{x} - t)]$ .

Then, substituting (7) into (5) leads to the master equation that determines the refractive indices and polarization vectors  $n$  and  $\delta \mathbf{E}$  for a given propagation direction  $\hat{\mathbf{n}}$ :

$$\epsilon_E \cdot \delta \mathbf{E} + \epsilon_B \cdot \mathbf{n} \times \delta \mathbf{E} + \mathbf{n} \times (\tilde{\mu}_B \cdot \mathbf{n} \times \delta \mathbf{E} + \tilde{\mu}_E \cdot \delta \mathbf{E}) = 0. \quad (8)$$

This equation can be written in matrix-vector form as

$$\Lambda \cdot \delta \mathbf{E} = (\epsilon_E + \tilde{\mathbf{n}} \cdot \tilde{\mu}_E + \epsilon_B \cdot \tilde{\mathbf{n}} + \tilde{\mathbf{n}} \cdot \tilde{\mu}_B \cdot \tilde{\mathbf{n}}) \cdot \delta \mathbf{E} = 0, \quad (9)$$

where  $\tilde{n}_{ij} = -\epsilon_{ijk} n_k$ .<sup>40</sup> Solving the equation involves two steps: solving  $\det(\Lambda(n)) = 0$  to find a value of  $n$  and then solving  $\Lambda(n) \cdot \delta \mathbf{E}(n) = 0$  to find  $\delta \mathbf{E}(n)$  associated with the value. Usually, multiple values of  $n$  are obtained, implying multirefringence. The relation between the polarization vectors associated with different values of  $n$  can hardly be known before finding the vectors explicitly, because they refer to different  $\Lambda$  matrices. But the relation between the polarization vectors of a given value of  $n$  may be known from the symmetry of  $\Lambda$ . To solve the equation, we need the explicit forms of  $\epsilon_E$ ,  $\epsilon_B$ ,  $\tilde{\mu}_B$ , and  $\tilde{\mu}_E$ , which are derived for Plebanski-type Lagrangians in Sec. IV A. From now on,  $\delta \mathbf{E}$  refers to the probe electric field without the factor  $\exp[i\omega(\mathbf{n} \cdot \mathbf{x} - t)]$ .

### III. COMPARISON WITH THE FORMULATION OF THE MAGNETO-ELECTRIC EFFECT IN MATERIALS SCIENCE

The 3+1 framework in Sec. II closely follows the approach in conventional optics and thus can facilitate direct comparison of the magneto-electric effects in field-modified vacua with those in optical media. Although magneto-electric effects are rarely observed in the usual optical media, some multiferroic materials have been known to exhibit these effects.<sup>18,19,41</sup> In this section, we compare the 3+1 framework with that used in materials science, in which the material response is found by minimizing the Helmholtz free energy  $\mathcal{F}$ .<sup>18</sup>

$$\mathcal{F}(\mathbf{E}, \mathbf{H}) = -\frac{1}{2} \mathbf{E} \cdot \epsilon \cdot \mathbf{E} - \frac{1}{2} \mathbf{H} \cdot \mu \cdot \mathbf{H} - \mathbf{E} \cdot \alpha \cdot \mathbf{H},$$

$$\mathbf{P} = -\frac{\partial \mathcal{F}}{\partial \mathbf{E}} = \epsilon \cdot \mathbf{E} + \alpha \cdot \mathbf{H}, \quad (10)$$

$$\mathbf{M} = -\frac{\partial \mathcal{F}}{\partial \mathbf{H}} = \mu \cdot \mathbf{H} + \alpha^T \cdot \mathbf{E},$$

where  $\epsilon$  and  $\mu$  (symmetric) are the permittivity and permeability tensors, respectively. The magneto-electric response tensor  $\alpha$  is odd under time reversal or space inversion to keep  $\mathcal{F}(\mathbf{E}, \mathbf{H})$  invariant under such transformations.<sup>35</sup> Here, all the quantities are real.

To compare (10) with (5), we express  $\delta \mathcal{P}$  and  $\delta \mathcal{M}$  in (4) and (5) in terms of  $\delta \mathbf{E}$  and  $\delta \mathbf{H}$ . Substituting  $\delta \mathbf{B} = \delta \mathbf{H} + \delta \mathcal{M}$  into (5) while assuming that  $\tilde{\mu}_B^{-1}$  exists, we obtain

$$\delta \mathcal{P} = (\epsilon_E - 1 - \epsilon_B \cdot \tilde{\mu}_B^{-1} \cdot \tilde{\mu}_E) \cdot \delta \mathbf{E} + \epsilon_B \cdot \tilde{\mu}_B^{-1} \cdot \delta \mathbf{H},$$

$$\delta \mathcal{M} = (\tilde{\mu}_B^{-1} - 1) \cdot \delta \mathbf{H} - \tilde{\mu}_B^{-1} \cdot \tilde{\mu}_E \cdot \delta \mathbf{E}. \quad (11)$$

For the magneto-electric parts in (11) to match those in (10),  $\epsilon$ ,  $\mu$ , and  $\alpha$  should be related to  $\epsilon_E$ ,  $\epsilon_B$ ,  $\tilde{\mu}_B$ , and  $\tilde{\mu}_E$  as follows:

$$\epsilon = \epsilon_E - 1 - \epsilon_B \cdot \tilde{\mu}_B^{-1} \cdot \tilde{\mu}_E, \quad \mu = \tilde{\mu}_B^{-1} - 1, \quad (12)$$

$$\alpha = \epsilon_B \cdot \tilde{\mu}_B^{-1} = -(\tilde{\mu}_B^{-1} \cdot \tilde{\mu}_E)^T.$$

These relations and the symmetry of  $\epsilon$  and  $\mu$  lead to the relations

$$\epsilon_E = \epsilon_E^T, \quad \tilde{\mu}_B = \tilde{\mu}_B^T, \quad \epsilon_B = -\tilde{\mu}_E^T, \quad (13)$$

which also hold for Plebanski-type Lagrangians, as will be shown in Sec. IV.

The relation reciprocal to (12) and (13) is obtained by expressing  $\mathbf{P}$  and  $\mathbf{M}$  in (10) in terms of  $\mathbf{E}$  and  $\mathbf{B}$ , which establishes the equivalence between the linear response coefficients in the materials science literature and those in our 3+1 formulation:

$$\epsilon_E = \epsilon - \alpha \cdot (1 + \mu)^{-1} \cdot \alpha^T + 1,$$

$$\epsilon_B = \alpha \cdot (1 + \mu)^{-1},$$

$$\tilde{\mu}_B = (1 + \mu)^{-1},$$

$$\tilde{\mu}_E = -(1 + \mu)^{-1} \cdot \alpha^T. \quad (14)$$

By substituting these expressions into (9) and rearranging terms, we obtain a factorized form of  $\Lambda$ :

$$\Lambda = 1 + \epsilon - (\tilde{\mathbf{n}} + \alpha) \cdot (1 + \mu)^{-1} \cdot (\tilde{\mathbf{n}} + \alpha)^T, \quad (15)$$

from which the property  $\Lambda^T = \Lambda$  is obvious. Because  $\Lambda$  is real-symmetric, the polarization vectors associated with a value of  $n$  can be made orthonormal to one another.<sup>42</sup> The factorization appears to be useful, but the specific forms of  $\epsilon$ ,  $\mu$ , and  $\alpha$  would be necessary for further analysis.

### IV. LIGHT PROPAGATION MODES IN NED OF PLEBANSKI-TYPE LAGRANGIANS

In this section, we derive the general expressions for the light propagation modes in NED described by Plebanski-type Lagrangians. The general framework of mode analysis in Sec. II is used to find the refractive indices and polarization vectors for a given propagation direction.

### A. Permittivity, permeability, and magneto-electric response tensors for Plebanski-type Lagrangians

When a strong electromagnetic field modifies the vacuum, the corresponding Lagrangian should be an analytic function of the Maxwell scalar  $F$  and Maxwell pseudoscalar  $G$  to satisfy Lorentz and gauge invariance. Such Lagrangians are called Plebanski-type Lagrangians:<sup>22</sup>

$$\mathcal{L} = \mathcal{L}(F, G) = -F + \mathcal{L}', \quad (16)$$

where  $\mathcal{L}'$  represents the non-Maxwell part. Typically,  $\mathcal{L}(F, G)$  is an even function of  $G$  to preserve parity.

In general, the Lagrangian may have an imaginary part, which leads to absorption of probe light.<sup>43</sup> For example, the HES Lagrangian has an imaginary part when either only an electric field exists or the electric field has a component parallel to the magnetic field. This imaginary part becomes significant as the purely electric field or the electric field component along the magnetic field approaches or surpasses the critical field strength  $E_c = m_e^2/e = 1.32 \times 10^{16}$  V/cm.<sup>44</sup> As a consequence, electron-positron pairs are produced by strong electromagnetic fields, in Sauter-Schwinger pair production or vacuum breakdown.<sup>8,11</sup> In the present work, we do not take such an imaginary part into account, to assume a stable medium. In practice, the imaginary term can be neglected when the electric field is weaker than  $E_c/3$  and the magnetic field.<sup>30</sup>

Once the Lagrangian is given, the polarization  $\mathcal{P}$  and magnetization  $\mathcal{M}$  are obtained by differentiating the Lagrangian with respect to the electric and magnetic fields:<sup>7,45</sup>

$$\mathbf{D} = \mathbf{E} + \mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}}, \quad \mathbf{H} = \mathbf{B} - \mathcal{M} = -\frac{\partial \mathcal{L}}{\partial \mathbf{B}}. \quad (17)$$

For Plebanski-type Lagrangians, we obtain

$$\mathbf{D} = -\mathcal{L}_F \mathbf{E} - \mathcal{L}_G \mathbf{B}, \quad \mathbf{H} = -\mathcal{L}_F \mathbf{B} + \mathcal{L}_G \mathbf{E}, \quad (18)$$

where the subscripts  $F$  and  $G$  denote partial differentiation of  $\mathcal{L}$  with respect to the respective quantities: for example,  $\mathcal{L}_{FG} \equiv \partial^2 \mathcal{L} / (\partial F \partial G)$ .

Then, we can obtain the permittivity, permeability, and magneto-electric response tensors by taking a variation of  $\mathbf{E}$  and  $\mathbf{B}$  in (18) and applying the decomposition argument in Sec. II:

$$\begin{aligned} \epsilon_E &= -L_F \mathbf{I} + [L_{FF} \mathbf{E}_0 \mathbf{E}_0 + L_{FG} (\mathbf{E}_0 \mathbf{B}_0 + \mathbf{B}_0 \mathbf{E}_0) + L_{GG} \mathbf{B}_0 \mathbf{B}_0], \\ \epsilon_B &= -L_G \mathbf{I} + [-L_{FF} \mathbf{E}_0 \mathbf{B}_0 + L_{FG} (\mathbf{E}_0 \mathbf{E}_0 - \mathbf{B}_0 \mathbf{B}_0) + L_{GG} \mathbf{B}_0 \mathbf{E}_0], \\ \bar{\mu}_B &= -L_F \mathbf{I} + [-L_{FF} \mathbf{B}_0 \mathbf{B}_0 + L_{FG} (\mathbf{B}_0 \mathbf{E}_0 + \mathbf{E}_0 \mathbf{B}_0) - L_{GG} \mathbf{E}_0 \mathbf{E}_0], \\ \bar{\mu}_E &= L_G \mathbf{I} + [L_{FF} \mathbf{B}_0 \mathbf{E}_0 + L_{FG} (\mathbf{B}_0 \mathbf{B}_0 - \mathbf{E}_0 \mathbf{E}_0) - L_{GG} \mathbf{E}_0 \mathbf{B}_0], \end{aligned} \quad (19)$$

where the products of two vectors denote dyadics. Here,  $L_F$ ,  $L_G$ ,  $L_{FF}$ ,  $L_{FG}$ , and  $L_{GG}$  denote  $\mathcal{L}_F$ ,  $\mathcal{L}_G$ ,  $\mathcal{L}_{FF}$ ,  $\mathcal{L}_{FG}$ , and  $\mathcal{L}_{GG}$  evaluated under condition  $(\mathbf{E}, \mathbf{B}) = (\mathbf{E}_0, \mathbf{B}_0)$ . From now on, we omit the subscript 0 so as not to clutter the notation. With these expressions for the linear response tensors, we can solve the master Eq. (9) to find the light modes, as will be shown in Sec. IV B. A similar formulation was presented by Robertson:<sup>46</sup> the modes were analyzed for weakly nonlinear Lagrangians and a fixed propagation direction, and the coupling to axions was considered.

The linear response tensors in (19) exhibit some symmetries. First,  $\epsilon_E = \epsilon_E^T$ ,  $\bar{\mu}_B = \bar{\mu}_B^T$ , and  $\epsilon_B = -\bar{\mu}_E^T$ , i.e., (13). Second,  $\epsilon_E \leftrightarrow \bar{\mu}_B$

and  $\epsilon_B \leftrightarrow -\bar{\mu}_E$  under the duality transformation  $(\mathbf{E}, \mathbf{B}) \rightarrow (-i\mathbf{B}, i\mathbf{E})$ , which keeps  $F$ ,  $G$ , and  $\mathcal{L}$  the same. Finally,  $\epsilon_E$  and  $\bar{\mu}_B$  have even parity, while  $\epsilon_B$  and  $\bar{\mu}_E$  have odd parity:  $\mathcal{L}$  should be an even function of  $G$ , i.e.,  $\mathcal{L}(F, -G) = \mathcal{L}(F, G)$ , to respect inversion symmetry.

### B. Light modes: Refractive indices and polarization vectors for an arbitrary propagation direction

In this subsection, we systematically solve the master Eq. (9) with the tensors (19) to find the light modes for an arbitrary propagation direction. The corresponding  $\Lambda$  in (9) is now given by

$$\Lambda = -L_F [(1 - n^2) \mathbf{I} + \mathbf{nn}] + L_{FF} \mathbf{PP} + L_{FG} (\mathbf{PQ} + \mathbf{QP}) + L_{GG} \mathbf{QQ}, \quad (20)$$

where

$$\mathbf{P} = \mathbf{E} + \mathbf{n} \times \mathbf{B}, \quad \mathbf{Q} = \mathbf{B} - \mathbf{n} \times \mathbf{E}. \quad (21)$$

From this dyadic expression for  $\Lambda$ , it is clear that  $\Lambda$  is real-symmetric as in the case of materials science, which is a feature of Plebanski-type Lagrangians. Through attempting to find complete squares of  $\mathbf{P}$  and  $\mathbf{Q}$  in  $\Lambda$ , we express  $\Lambda$  as a linear combination of the identity matrix and self-conjugate dyadics, i.e., dyadics of two identical vectors:<sup>47</sup>

$$\Lambda = \mathbf{XX} + \mathbf{YY} + \mathbf{ZZ} + \mathbf{U}, \quad (22)$$

where

$$\mathbf{X} = a\mathbf{P} + b\mathbf{Q}, \quad \mathbf{Y} = c\mathbf{Q}, \quad \mathbf{Z} = d\mathbf{n}, \quad \mathbf{U} = e\mathbf{I}, \quad (23)$$

and

$$\begin{aligned} a^2 &= L_{FF}, & ab &= L_{FG}, & b^2 &= \frac{L_{FG}^2}{L_{FF}}, \\ c^2 &= \frac{L_{FF} L_{GG} - L_{FG}^2}{L_{FF}}, & d^2 &= -L_F, \\ e &= -L_F (1 - n^2). \end{aligned} \quad (24)$$

The form of  $\Lambda$  in (22) implies that the reciprocal vectors of  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  may be used as basis vectors to represent  $\delta \mathbf{E}$ , as in crystallography. However, this requires nondegeneracy, i.e., the condition that the volume  $V := \mathbf{X} \times \mathbf{Y} \cdot \mathbf{Z}$  is not zero. The degeneracy ( $V = 0$ ) is caused by either  $acd = 0$  or  $\mathbf{P} \times \mathbf{Q} \cdot \hat{\mathbf{n}} = 0$ , as can be found from (21) and (23). The first condition  $acd = 0$  is independent of  $\hat{\mathbf{n}}$  and depends only on the functional form of  $\mathcal{L}(F, G)$  and the background field. In fact, the condition reduces to  $c = 0$ , because  $a^2 = L_{FF}$  and  $d^2 = -L_F$  do not vanish for nontrivial Lagrangians. An example of having  $c = 0$  is the MM Lagrangian, as will be shown in Sec. V C. By contrast, the second condition  $\mathbf{P} \times \mathbf{Q} \cdot \hat{\mathbf{n}} = 0$  depends also on  $\hat{\mathbf{n}}$ . The second condition is met by a configuration satisfying  $\mathbf{B}_t = \hat{\mathbf{n}} \times \mathbf{E}_t$ ,<sup>48</sup> which we call the free-propagation configuration (FPC). In the FPC,  $n$  is unity, and  $\mathbf{S} := \mathbf{E} \times \mathbf{B}$  heads along  $\hat{\mathbf{n}}$ , and any vector perpendicular to  $\hat{\mathbf{n}}$  is a polarization vector. In other words, the light in FPC propagates as if it were in a free vacuum. In the nondegenerate case, the FPC is the only configuration with  $n = 1$ . In the degenerate case of  $c = 0$ , however, another configuration has  $n = 1$ , as shown in Sec. IV B 2. A special case of FPC is the collinear configuration:  $\mathbf{E} \parallel \mathbf{B} \parallel \hat{\mathbf{n}}$ . Below, we deal with the nondegenerate case of  $c \neq 0$  first and

then the degenerate case of  $c = 0$ , disregarding the FPC, owing to its triviality.

### 1. Nondegenerate case ( $c \neq 0$ )

Nondegeneracy ( $V \neq 0$ ) allows us to represent  $\delta\mathbf{E}$  as a linear combination of the reciprocal vectors of  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$ :

$$\delta\mathbf{E} = p\mathbf{X}' + q\mathbf{Y}' + r\mathbf{Z}', \quad (25)$$

where

$$\mathbf{X}' = \frac{\mathbf{Y} \times \mathbf{Z}}{V}, \quad \mathbf{Y}' = \frac{\mathbf{Z} \times \mathbf{X}}{V}, \quad \mathbf{Z}' = \frac{\mathbf{X} \times \mathbf{Y}}{V}. \quad (26)$$

The linear equations for  $p$ ,  $q$ , and  $r$  are then obtained by multiplying (inner product) each of  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  from the left into the master equation  $\mathbf{\Lambda} \cdot \delta\mathbf{E} = 0$  with  $\mathbf{\Lambda}$  given by (22):

$$\begin{aligned} Mp + Hq &= 0, & Hp + Nq &= 0, \\ r &= -\frac{\mathbf{Z} \cdot \mathbf{X}}{d^2}p - \frac{\mathbf{Z} \cdot \mathbf{Y}}{d^2}q, \end{aligned} \quad (27)$$

where

$$\begin{aligned} H &= \left[ \mathbf{X} \cdot \mathbf{Y} - \frac{(\mathbf{Z} \cdot \mathbf{X})(\mathbf{Z} \cdot \mathbf{Y})}{d^2} \right], \\ M &= \left[ X^2 + e - \frac{(\mathbf{Z} \cdot \mathbf{X})^2}{d^2} \right], \\ N &= \left[ Y^2 + e - \frac{(\mathbf{Z} \cdot \mathbf{Y})^2}{d^2} \right]. \end{aligned} \quad (28)$$

It is assumed that  $d^2 = -L_F \neq 0$ , which holds in general because any nonlinear Lagrangian should have the Maxwell term, i.e.,  $-F$ , as the leading term.

The linear equations in  $p$ ,  $q$ , and  $r$  (27) are solved in two steps. In the first step, the null determinant condition,  $MN - H^2 = 0$ , is solved for  $n$ . In the second step, each  $n$  is substituted into (27), which is then solved for  $(p, q, r)$ , shown in Table I. In the particular case of  $H = M = N = 0$ ,  $(p, q)$  can have arbitrary values, and the resulting polarization vector spans a plane:

$$\delta\mathbf{E} = p\left(\mathbf{X}' - \frac{\mathbf{X} \cdot \mathbf{Z}}{d^2}\mathbf{Z}'\right) + q\left(\mathbf{Y}' - \frac{\mathbf{Y} \cdot \mathbf{Z}}{d^2}\mathbf{Z}'\right). \quad (29)$$

In contrast to the photon in a free vacuum, a photon in the NED of a Plebanski-type Lagrangian acquires a longitudinal component when  $r \neq 0$ :  $\delta\mathbf{E} \cdot \mathbf{n} = r/d$  from (23), (25), and (26). Such a longitudinal component appears in anisotropic dielectric media.<sup>38</sup>

Now we solve  $MN - H^2 = 0$  for the refractive indices. Factorizing  $MN - H^2$  by using (21), (23), (24), and (28), we obtain the Fresnel equation:

$$\left( \frac{\tilde{P}^2}{n^2 - 1} - \lambda^{(+)} \right) \left( \frac{\tilde{P}^2}{n^2 - 1} - \lambda^{(-)} \right) = 0, \quad (30)$$

where

$$\begin{aligned} \tilde{P}^2 &:= P^2 - n^2 P_n^2 = E^2 - 2nS_n + n^2(B_t^2 - E_n^2), \\ \lambda^{(\pm)} &= \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma\alpha}}{2\gamma}. \end{aligned} \quad (31)$$

**TABLE I.** Determination of polarization vectors for a given refractive index. The symbol  $\sim$  between two quantities indicates that the two vectors are equivalent, i.e., differ only by a factor. In this table, the case  $n = 1$  is excluded. In the nondegenerate case,  $n = 1$  leads to the FPC. In the degenerate case,  $n = 1$  can hold when  $\delta\mathbf{E} \propto \mathbf{Z} \times \mathbf{X}$  for general configurations as well as the FPC.

| Nondegenerate case ( $c \neq 0$ )<br>$n (\neq 1)$ from $MN - H^2 = 0$ (33)                   | $\delta\mathbf{E} = p\mathbf{X}' + q\mathbf{Y}' + r\mathbf{Z}'$<br>$\mathbf{X}', \mathbf{Y}', \mathbf{Z}'$ from (26)                                                           |
|----------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Cases                                                                                        | $(p, q, r)$                                                                                                                                                                    |
| $H \neq 0, M \neq 0, N \neq 0$                                                               | $(H, -M, -(\mathbf{Z} \cdot \mathbf{X})H/d^2 + (\mathbf{Z} \cdot \mathbf{Y})M/d^2)$<br>$\sim (-N, H, (\mathbf{Z} \cdot \mathbf{X})N/d^2 - (\mathbf{Z} \cdot \mathbf{Y})H/d^2)$ |
| $H = 0, M \neq 0, N = 0$                                                                     | $(0, -M, (\mathbf{Z} \cdot \mathbf{Y})M/d^2) \sim (0, 1, -(\mathbf{Z} \cdot \mathbf{Y})/d^2)$                                                                                  |
| $H = 0, M = 0, N \neq 0$                                                                     | $(-N, 0, (\mathbf{Z} \cdot \mathbf{X})N/d^2) \sim (1, 0, -(\mathbf{Z} \cdot \mathbf{X})/d^2)$                                                                                  |
| $H = 0, M = 0, N = 0$                                                                        | $(p, q, -(\mathbf{Z} \cdot \mathbf{X})p/d^2 - (\mathbf{Z} \cdot \mathbf{Y})q/d^2);$<br>$p$ and $q$ are arbitrary.                                                              |
| Degenerate case ( $c = 0$ )<br>$n (\neq 1)$ from $\tilde{M}\tilde{N} - \tilde{H}^2 = 0$ (38) | $\delta\mathbf{E} = p\mathbf{X}' + r\mathbf{Z}'$<br>$\mathbf{X}', \mathbf{Z}'$ from (36)                                                                                       |
| Cases                                                                                        | $(p, r)$                                                                                                                                                                       |
| $\tilde{H} \neq 0, \tilde{M} \neq 0, \tilde{N} \neq 0$                                       | $(\tilde{H}, -\tilde{M}) \sim (-\tilde{N}, \tilde{H})$                                                                                                                         |
| $\tilde{H} = 0, \tilde{M} \neq 0, \tilde{N} = 0$                                             | $(0, -\tilde{M}) \sim (0, 1)$                                                                                                                                                  |
| $\tilde{H} = 0, \tilde{M} = 0, \tilde{N} \neq 0$                                             | $(-\tilde{N}, 0) \sim (1, 0)$                                                                                                                                                  |
| $\tilde{H} = 0, \tilde{M} = 0, \tilde{N} = 0$                                                | $(p, r); p$ and $r$ are arbitrary.                                                                                                                                             |



The parameters  $\alpha$ ,  $\beta$ , and  $\gamma$  are determined solely by the background field, regardless of  $\hat{\mathbf{n}}$ , and thus so is  $\lambda^{(\pm)}$ :

$$\begin{aligned}\alpha &= L_F^2 + 2L_F(GL_{FG} - FL_{GG}) - (L_{FF}L_{GG} - L_{FG}^2)G^2 \\ &= d^4 - 2d^2[abG - (b^2 + c^2)F] - a^2c^2G^2, \\ \beta &= L_F(L_{FF} + L_{GG}) - 2F(L_{FF}L_{GG} - L_{FG}^2) \\ &= -d^2(a^2 + b^2 + c^2) - 2a^2c^2F, \\ \gamma &= L_{FF}L_{GG} - L_{FG}^2 = a^2c^2.\end{aligned}\quad (32)$$

Equation (30) can be rewritten as

$$(B_t^2 - E_n^2 - \lambda^{(\pm)})n^2 - 2S_n n + E_n^2 + \lambda^{(\pm)} = 0, \quad (33)$$

from which we obtain the refractive indices:

$$n_{\pm}^{(i)} = \frac{-S_n \pm \sqrt{S_n^2 + (\lambda^{(i)} - B_t^2 + E_n^2)(\lambda^{(i)} + E^2)}}{\lambda^{(i)} - B_t^2 + E_n^2}, \quad (34)$$

where  $i = \pm$ . Without loss of generality, we take only positive values of  $n$  and solve (27) to obtain the corresponding polarization vectors (see Table I). It should be mentioned that we may have multirefringence, even beyond birefringence, because (30) is a quartic equation in  $n$ .<sup>49</sup> For a purely magnetic background field,  $(n^{(i)})^2$  has the simpler form

$$(n^{(i)})^2 = \left| \frac{\lambda^{(i)}}{\lambda^{(i)} - B_t^2} \right|, \quad (35)$$

implying birefringence.

So far, we have implicitly assumed that  $n \neq 1$ . When  $n = 1$ , the dyadic  $\Lambda$  in (22) becomes  $\Lambda = \mathbf{XX} + \mathbf{YY} + \mathbf{ZZ}$  because  $e = 0$ . Then, the condition  $MN - H^2 = 0$  reduces to  $\mathbf{P}_t = \mathbf{Q}_t = 0$ , which is satisfied only for the FPC. In such a case, the light sees no effect of NED.

## 2. Degenerate case ( $c = 0$ )

The condition  $c = 0$  nullifies  $\mathbf{Y}$  in (23) and thus simplifies  $\Lambda$ . Unless  $\mathbf{Z} \parallel \mathbf{X}$ , which turns out to be the FPC, we can proceed similarly to the nondegenerate case by introducing two-dimensional reciprocal vectors:

$$\mathbf{X}' = \frac{Z^2 \mathbf{X} - (\mathbf{X} \cdot \mathbf{Z})\mathbf{Z}}{Z^2 X^2 - (\mathbf{X} \cdot \mathbf{Z})^2}, \quad \mathbf{Z}' = \frac{X^2 \mathbf{Z} - (\mathbf{Z} \cdot \mathbf{X})\mathbf{X}}{X^2 Z^2 - (\mathbf{Z} \cdot \mathbf{X})^2}, \quad (36)$$

where  $\mathbf{X}'$  and  $\mathbf{Z}'$  span the same plane spanned by  $\mathbf{X}$  and  $\mathbf{Z}$ ;  $\delta\mathbf{E}$  has no component perpendicular to the plane. The counterparts to (25), (27), and (28) become as follows:

$$\begin{aligned}\delta\mathbf{E} &= p\mathbf{X}' + r\mathbf{Z}', \\ \tilde{M}p + \tilde{H}r &= 0, \quad \tilde{H}p + \tilde{N}r = 0, \\ \tilde{H} &= \mathbf{X} \cdot \mathbf{Z}, \quad \tilde{N} = Z^2 + e, \quad \tilde{M} = X^2 + e.\end{aligned}\quad (37)$$

The null determinant condition  $\tilde{M}\tilde{N} - \tilde{H}^2 = 0$  is given as a quadratic equation in  $n$ :

$$\begin{aligned}[a^2(E_n^2 - B_t^2) + b^2(B_n^2 - E_t^2) - L_F - 2abG]n^2 \\ + 2S_n(a^2 + b^2)n - a^2E^2 + 2abG - b^2B^2 + L_F = 0.\end{aligned}\quad (38)$$

Once  $n$  is available from this equation, the polarization vectors can be found from  $(p, r)$  in Table I.

In contrast to the nondegenerate case, the degenerate case always has a non-FPC with  $n = 1$ . When  $n = 1$ ,  $\Lambda = \mathbf{XX} + \mathbf{ZZ}$ . When  $\mathbf{Z} \parallel \mathbf{X}$  (non-FPC),  $\mathbf{Z} \times \mathbf{X}$  becomes the polarization vector, because  $\Lambda \cdot \delta\mathbf{E} = 0$  is satisfied by construction. The case  $\mathbf{Z} \parallel \mathbf{X}$  corresponds to the FPC. An advantage of our 3+1 formulation is a transparent analysis of the degenerate case compared with that of the covariant formulation.

## C. Procedure to find modes in nondegenerate NED

The procedure to find light modes in nondegenerate nonlinear electrodynamics is outlined in Table II. Here, we succinctly present the essential formulas used for the procedure in order. It is assumed that the formulas for  $\{L_F, L_{FF}, L_{FG}, L_{GG}\}$ , and thus  $\{a^2, ab, b^2, c^2, d^2\}$  in (24) are known, regardless of the field configuration.

**TABLE II.** Procedure to find light modes in a nonlinear nondegenerate vacuum. The steps from 7 to 9 should be implemented for each value of  $n$  found in step 6. In Sec. IV C, we present the essential formulas used in the procedure in order. A similar procedure can be set up in the degenerate case.

| Step | Task                                                                                                     | Equations     |
|------|----------------------------------------------------------------------------------------------------------|---------------|
| 1    | Specify Lagrangian                                                                                       | (16)          |
| 2    | Calculate $a^2, ab, b^2, c^2$ , and $d^2$ without specifying $\mathbf{E}$ and $\mathbf{B}$               | (24)          |
| 3    | Specify $\mathbf{E}$ , $\mathbf{B}$ , and $\hat{\mathbf{n}}$ and rule out the FPC                        |               |
| 4    | Calculate $a^2, ab, b^2, c^2$ , and $d^2$ for $\mathbf{E}$ and $\mathbf{B}$ , and confirm $c \neq 0$     | (24)          |
| 5    | Calculate $\alpha, \beta$ , and $\gamma$ in terms of $a^2, ab, b^2, c^2$ , and $d^2$                     | (32)          |
| 6    | Calculate $\lambda^{(\pm)}$ and $n > 0$                                                                  | (31) and (34) |
| 7    | Calculate $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{V}, \mathbf{X}', \mathbf{Y}'$ , and $\mathbf{Z}'$ | (23) and (26) |
| 8    | Calculate $H, M$ , and $N$                                                                               | (28)          |
| 9    | Determine $(p, q, r)$ and calculate $\delta\mathbf{E}$                                                   | Table I, (25) |

Given a configuration specified by  $\{\mathbf{E}, \mathbf{B}, \hat{\mathbf{n}}\}$  in the 3+1 formulation, we then specialize  $\{a^2, ab, b^2, c^2, d^2\}$  to the configuration and calculate  $\{\alpha, \beta, \gamma\}$  in (32) to obtain the refractive index  $n$  from (34). The refractive index may need to be numerically evaluated if the Lagrangian has a complicated functional form. With the refractive index, we calculate  $\mathbf{P}$  and  $\mathbf{Q}$  in (21) and the following auxiliary parameters:

$$\begin{aligned} \mathbf{P}_c &= \mathbf{E}_c + n\mathbf{B}_t, & \mathbf{Q}_c &= \mathbf{B}_c - n\mathbf{E}_t, \\ \mathbf{P} \times \mathbf{Q} &= \mathbf{S} - n(E^2 + B^2)\hat{\mathbf{n}} + n(E_n\mathbf{E} + B_n\mathbf{B}) + n^2 S_n \hat{\mathbf{n}}, \\ \widetilde{PQ} &:= \mathbf{P} \cdot \mathbf{Q} - n^2 P_n Q_n = -(1 - n^2)G, \\ \widetilde{P}^2 &:= P^2 - n^2 P_n^2 = E^2 - 2nS_n + n^2(B_t^2 - E_n^2), \\ \widetilde{Q}^2 &:= Q^2 - n^2 Q_n^2 = B^2 - 2nS_n + n^2(E_t^2 - B_n^2), \end{aligned} \quad (39)$$

where the subscripts  $n$  and  $t$  denote components respectively parallel and transverse to  $\hat{\mathbf{n}}$ .

Then,  $acH$ ,  $M$ , and  $N$  in (28) are expressed as

$$\begin{aligned} acH &= a^2 c^2 \widetilde{PQ} + abc^2 \widetilde{Q}^2, \\ M &= a^2 \widetilde{P}^2 + 2ab\widetilde{PQ} + b^2 \widetilde{Q}^2 + d^2(1 - n^2), \\ N &= c^2 \widetilde{Q}^2 + d^2(1 - n^2). \end{aligned} \quad (40)$$

Unless  $acH = M = N = 0$ , we form the following two vectors proportional to the polarization vectors in Table I:

$$\begin{aligned} \delta\mathbf{E}_1 &= a^2 M \mathbf{P}_c + (acH + abM)\mathbf{Q}_c \\ &\quad - \frac{(a^2 P_n + abQ_n)acH - a^2 c^2 M Q_n}{d^2} \mathbf{P} \times \mathbf{Q}, \\ \delta\mathbf{E}_2 &= -\frac{acH a^2}{c^2} \mathbf{P}_c - \left(a^2 N + \frac{acH ab}{c^2}\right) \mathbf{Q}_c \\ &\quad + \frac{a^2[(a^2 P_n + abQ_n)N - acH Q_n]}{d^2} \mathbf{P} \times \mathbf{Q}. \end{aligned} \quad (41)$$

We can choose any nonzero vector from  $\delta\mathbf{E}_1$  and  $\delta\mathbf{E}_2$  as the polarization vector for the refractive index  $n$ . When both are nonzero, they differ from each other only by a multiplicative factor. When  $acH = M = N = 0$ , the polarization vector spans a plane:

$$\begin{aligned} \delta\mathbf{E} &= [d^2 \mathbf{Q}_c - (a^2 P_n + abQ_n)\mathbf{P} \times \mathbf{Q}]p \\ &\quad + (a^2 d^2 \mathbf{P}_c + abd^2 \mathbf{Q}_c + a^2 c^2 Q_n \mathbf{P} \times \mathbf{Q})q, \end{aligned} \quad (42)$$

where  $p$  and  $q$  are arbitrary real numbers.

In summary, light modes are completely determined by the formulas (32), (34), and (40)–(42) once  $\{a^2, ab, b^2, c^2, d^2\}$  (24) and the configuration parameters (39) are known.

#### D. Comparison with covariant formulation

The most economical way to obtain light modes is the covariant four-vector formulation, to which our 3+1 formulation is equivalent in this regard. With the photon propagation four-vector  $k^\mu = \omega(1, \mathbf{n})$ , the covariant vectors used for polarization tensors take the form

$$F^{\mu\nu} k_\nu = \omega(\mathbf{E} \cdot \mathbf{n}, \mathbf{P}), \quad F^{*\mu\nu} k_\nu = \omega(\mathbf{B} \cdot \mathbf{n}, \mathbf{Q}). \quad (43)$$

The Fresnel equation in the covariant formulation, for instance Eq. (12) for  $f^2 := F^{\alpha\mu} F_\alpha{}^\nu k_\mu k_\nu$  in De Lorenci *et al.*<sup>49</sup> recovers (30) with the same  $\{\alpha, \beta, \gamma\}$ .

However, the 3+1 formulation has a more direct connection to the usual concepts in optics, such as the permittivity, permeability, and magneto-electric response tensors (19), and provides the refractive indices and polarization vectors in terms of familiar quantities  $\mathbf{E}, \mathbf{B}$ , and  $\hat{\mathbf{n}}$ . Thus, it can allow a richer comparison of the nonlinear vacua with conventional optical media. The formulation also reveals the unique mathematical structure of the nonlinear vacuum's response: the Fresnel matrix  $\Lambda$  consists of self-conjugate dyadics (22), which facilitates the construction of polarization vectors. Finally, the formulation is related to the 3+1 formulation in curved spacetime,<sup>32</sup> which is popular in numerical simulation of general relativistic phenomena.<sup>50</sup> In our 3+1 formulation in Minkowski spacetime, we use the Killing vector  $\partial_t$  to decompose the spacetime into one-parameter spacelike hypersurfaces  $\Sigma_t$ . Each observer on  $\Sigma_t$  measures the energy-momentum of photons as  $k^\mu$  and the electromagnetic field  $\mathbf{E}, \mathbf{B}$ .

#### V. APPLICATION TO VARIOUS NED LAGRANGIANS

In this section, we apply the 3+1 formulation to several NED Lagrangians for the case of a purely magnetic background field: post-Maxwellian (PM),<sup>28</sup> Born-Infeld (BI),<sup>5</sup> ModMax (MM),<sup>51,52</sup> and Heisenberg-Euler-Schwinger (HES) QED<sup>53</sup> Lagrangians. The PM Lagrangians are frequently used in X-ray polarimetry for light propagation near pulsars and also in black hole physics. Recently, these NED Lagrangians and Plebanski-type Lagrangians, in general, have been actively used for theoretical black holes.<sup>54</sup> Theoretical magnetic black holes whose magnetic field strength is at electroweak scales have been proposed,<sup>55</sup> and these require QED corrections in the context of general relativity.

Without loss of generality, we assume that the magnetic field is along the  $z$  axis, and the light's propagation direction vector lies on the  $xz$  plane:  $\mathbf{B} = B_0 \hat{\mathbf{z}}$  and  $\hat{\mathbf{n}} = (\sin \theta, 0, \cos \theta)$ , as shown in Fig. 1. The light modes for a given Lagrangian, i.e.,  $n$  and  $\delta\mathbf{E}$ , are found by

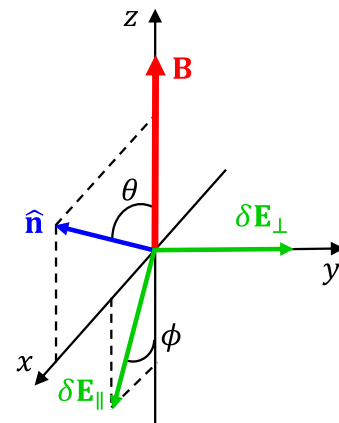


FIG. 1. Configuration of the background magnetic field  $\mathbf{B} = B_0 \hat{\mathbf{z}}$ , the light propagation direction  $\hat{\mathbf{n}}$ , and the two polarization vectors  $\delta\mathbf{E}_\perp$  and  $\delta\mathbf{E}_\parallel$ . The vector  $\delta\mathbf{E}_\perp$  is perpendicular to the plane formed by  $\mathbf{B}$  and  $\hat{\mathbf{n}}$ , while  $\delta\mathbf{E}_\parallel$  is in the plane. The light modes in Table III refer to this configuration.



**TABLE III.** Refractive indices and polarization vectors for various Lagrangians in a purely magnetic background field:  $\mathbf{B} = B_0 \hat{z}$  and  $\hat{\mathbf{n}} = (\sin \theta, 0, \cos \theta)$ , as shown in Fig. 1. For HES,  $a^2$ ,  $c^2$ , and  $d^2$  are from (55). The FPC is not included here.

| Lagrangian                       | Degeneracy ( $c = 0$ ) | (Refractive index) <sup>2</sup>                                                                                            | Polarization vector                                                                                                                        | Parameters                                                                                        |
|----------------------------------|------------------------|----------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------|
| Post-Maxwellian (PM)             | No                     | $n_{\perp}^2 \equiv \frac{1}{1-\mu \sin^2 \theta}$<br>$n_{\parallel}^2 \equiv \frac{1+\epsilon}{1+\epsilon \cos^2 \theta}$ | $\delta \mathbf{E}_{\perp} \equiv (0, 1, 0)$<br>$\delta \mathbf{E}_{\parallel} \equiv (\cos \theta, 0, -\frac{1}{1+\epsilon} \sin \theta)$ | $\mu = \frac{2\eta_1 B_0^2}{1-\eta_1 B_0^2}$<br>$\epsilon = \frac{2\eta_2 B_0^2}{1-\eta_1 B_0^2}$ |
| Born-Infeld (BI)                 | No                     | $n_{\parallel}^2$                                                                                                          | $p\delta \mathbf{E}_{\perp} + q\delta \mathbf{E}_{\parallel}$                                                                              | $\epsilon = B_0^2/T$<br>$p, q$ arbitrary                                                          |
| ModMax (MM)                      | Yes                    | $n_{\perp}^2 (= 1)$<br>$n_{\parallel}^2$                                                                                   | $\delta \mathbf{E}_{\perp}$<br>$\delta \mathbf{E}_{\parallel}$                                                                             | $\mu = 0$<br>$\epsilon = e^{2g} - 1$                                                              |
| Heisenberg-Euler-Schwinger (HES) | No                     | $n_{\perp}^2$<br>$n_{\parallel}^2$                                                                                         | $\delta \mathbf{E}_{\perp}$<br>$\delta \mathbf{E}_{\parallel}$                                                                             | $\mu = B_0^2 a^2 / d^2$<br>$\epsilon = B_0^2 c^2 / d^2$                                           |

following the procedure in Table II and Sec. IV C, as summarized in Table III.

Interestingly, the light propagation modes of the four Lagrangians exhibit a common structure, as manifestly parameterized in Table III, owing to the fact that these Lagrangians are of Plebanski type. The polarization vector is either perpendicular to the plane ( $\perp$ ) formed by the magnetic field and the propagation direction or on the plane ( $\parallel$ ). As a consequence of the common structure, the photon propagation at special angles is universal. When  $\theta = 0$ ,  $n_{\perp} = n_{\parallel} = 1$  with  $\delta \mathbf{E}_{\parallel} = (1, 0, 0)$ : a photon propagating along the magnetic field sees no background field effect. Note that  $\delta \mathbf{E}_{\perp} = (0, 1, 0)$  regardless of  $\theta$ . When  $\theta = \pi/2$ ,  $\delta \mathbf{E}_{\parallel} = (0, 0, 1)$ : a photon propagating perpendicular to the magnetic field has no longitudinal component. In general, nonzero  $\epsilon$  implies the presence of a longitudinal component:  $\delta \mathbf{E}_{\parallel} \cdot \hat{\mathbf{n}} = (\sin \theta \cos \theta) \epsilon / (1 + \epsilon)$ . The details specific to each Lagrangian are discussed below.

### A. Post-Maxwellian Lagrangian

The PM Lagrangian<sup>28</sup> is given by

$$\mathcal{L}_{\text{PM}} = -F + \eta_1 F^2 + \eta_2 G^2, \quad (44)$$

where  $\eta_1$  and  $\eta_2$  are parameters. It is useful as the generic weak-field form of the Plebanski Lagrangians that approaches the Maxwell Lagrangian ( $-F$ ) as the field vanishes. For example,  $\eta_1 = \eta_2 = 1/(2T)$  and  $\eta_1/4 = \eta_2/7 = e^4/(360\pi^2 m^4)$  give the BI and HES<sup>7,8,56</sup> Lagrangians in the weak-field limit, respectively. The PM Lagrangian has been widely used to design vacuum birefringence experiments, since only subcritical fields are available in the laboratory, even with the most intense lasers.<sup>44,57,58</sup> One may extend  $\mathcal{L}_{\text{PM}}$  by adding a product of  $F$  and  $G$ : a parity-violating term  $\eta_3 FG$  or a parity-conserving term  $\eta_3 F\sqrt{G^2}$ .<sup>28</sup>

The procedure in Table II yields the coefficients (24) and (32), regardless of the specific configuration:

$$\begin{aligned} a^2 &= 2\eta_1, & ab &= 0, & b^2 &= 0, \\ c^2 &= 2\eta_2, & d^2 &= 1 - 2F\eta_1, \\ \alpha &= 4F^2\eta_1^2 - 4(2F^2 + G^2)\eta_1\eta_2 - 4F(\eta_1 - \eta_2) + 1, \\ \beta &= 4F\eta_1^2 - 4F\eta_1\eta_2 - 2(\eta_1 + \eta_2), & \gamma &= 4\eta_1\eta_2. \end{aligned} \quad (45)$$

The light modes in the presence of a purely magnetic background field are given in Table III, and they reduce in the weak-field limit to the well-known modes  $n_{\perp} = 1 + \eta_1 B_0^2 \sin^2 \theta$ , with  $\delta \mathbf{E}_{\perp} = (0, 1, 0)$ , and  $n_{\parallel} = 1 + \eta_2 B_0^2 \sin^2 \theta$ , with  $\delta \mathbf{E}_{\parallel} = (\cos \theta, 0, -\sin \theta)$ .<sup>21</sup> The parallel mode has a longitudinal component:  $\delta \mathbf{E}_{\parallel} \cdot \hat{\mathbf{n}} = \eta_2 B_0^2 \sin 2\theta$ .

### B. Born-Infeld Lagrangian

The BI Lagrangian, introduced to remove the self-energy divergence in classical electrodynamics, is given by<sup>5</sup>

$$\mathcal{L}_{\text{BI}} = T - \sqrt{T^2 + 2FT - G^2}, \quad (46)$$

where  $T$  is a parameter. In the weak-field limit ( $T \gg F, G$ ), the BI Lagrangian becomes the PM Lagrangian with  $\eta_1 = \eta_2 = 1/(2T)$ .

The coefficients (24), (31), and (32) are given by

$$\begin{aligned} a^2 &= \frac{T^2}{D^{3/2}}, & ab &= -\frac{GT}{D^{3/2}}, & b^2 &= \frac{G^2}{D^{3/2}}, \\ c^2 &= \frac{1}{D^{1/2}}, & d^2 &= \frac{T}{D^{1/2}}, \\ \alpha &= \frac{T^2(2F+T)^2}{D^2}, & \beta &= -\frac{2T^2(2F+T)}{D^2}, & \gamma &= \frac{T^2}{D^2}, \\ \lambda^{(\pm)} &= 2F + T, \end{aligned} \quad (47)$$

where

$$D = 2FT - G^2.$$

As shown in Table III, the BI Lagrangian does not have birefringence in a pure magnetic field.<sup>52</sup> Furthermore,  $H = M = N = 0$  holds (see Table I), and thus the polarization vector spans a plane as in a

free vacuum, but possibly with a nonzero longitudinal component:  $\delta \mathbf{E} \cdot \hat{\mathbf{n}} = q\delta \mathbf{E}_{\parallel} \cdot \hat{\mathbf{n}} = qB_0 \sin 2\theta/(2T)$ .

### C. ModMax Lagrangian

The MM Lagrangian, introduced as a modification of the Maxwell theory to preserve duality invariance and conformal invariance,<sup>51</sup> is given by

$$\mathcal{L}_{\text{MM}} = -F \cosh g + \sqrt{F^2 + G^2} \sinh g, \quad (48)$$

where  $g \geq 0$  for a lightlike or subluminal propagation of light. It reduces to the Maxwell Lagrangian when  $g = 0$ .

The coefficients (24) are found as

$$\begin{aligned} a^2 &= \frac{G^2}{R^3} \sinh g, & ab &= -\frac{FG}{R^3} \sinh g, & b^2 &= \frac{F^2}{R^3} \sinh g, \\ c^2 &= 0, & d^2 &= \cosh g - \frac{F}{R} \sinh g, \end{aligned} \quad (49)$$

where  $R = \sqrt{F^2 + G^2}$ . As  $c^2 = 0$  (a consequence of the conformal and duality invariance), the MM Lagrangian is degenerate, and the formulas in Sec. IV B 2 are used for analysis.

The light modes in a pure magnetic field are shown in Table III. In contrast to the modes for other Lagrangians, the refractive indices and polarization vectors are independent of the strength of the background magnetic field, which is a consequence of conformal invariance.<sup>52</sup> The perpendicular mode always has  $n_{\perp} = 1$ , regardless of the configuration, as discussed in Sec. IV B 2. A photon in this mode propagates as if it sees no background field.

### D. Heisenberg-Euler-Schwinger Lagrangian

Heisenberg, Euler, and Schwinger obtained the one-loop correction to the Maxwell Lagrangian due to spin-1/2 fermions of mass  $m$  in constant electric and magnetic fields of arbitrary strengths.<sup>7,8</sup> The exact one-loop action is given as a proper-time integral:

$$\begin{aligned} \mathcal{L}^{(1)}(h, g) &= -\frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2 s}}{s^3} \left\{ ehs \coth(esh) \operatorname{egs} \cot(egs) \right. \\ &\quad \left. - \left[ 1 + \frac{(esh)^2 - (egs)^2}{3} \right] \right\}, \end{aligned} \quad (50)$$

where

$$h = \sqrt{\sqrt{F^2 + G^2} + F}, \quad g = \sqrt{\sqrt{F^2 + G^2} - F}. \quad (51)$$

The HES Lagrangian is obtained by adding the correction to the Maxwell Lagrangian:

$$\mathcal{L}_{\text{HES}}(h, g) = \frac{g^2 - h^2}{2} + \mathcal{L}^{(1)}(h, g). \quad (52)$$

The direct analytical form of  $\mathcal{L}_{\text{HES}}$  for arbitrary constant electromagnetic fields is not yet known. However, the direct form for the case of  $G = 0$  was obtained using dimensional regularization,<sup>53</sup> and it was confirmed by gamma-function regularization in the in-out formalism.<sup>59</sup>

$$\begin{aligned} \mathcal{L}_{\text{HES}}^{G=0}(\bar{h}) &= \frac{m^4}{8\pi^2 \bar{h}^2} \left[ -\frac{\pi}{4\alpha_e} + \zeta'(-1, \bar{h}) \right] \\ &\quad + \frac{m^4}{8\pi^2 \bar{h}^2} \left[ -\frac{1}{12} + \frac{\bar{h}^2}{4} - \left( \frac{1}{12} - \frac{\bar{h}}{2} + \frac{\bar{h}^2}{2} \right) \ln \bar{h} \right], \end{aligned} \quad (53)$$

where  $\bar{h} = m^2/(2eh)$ ,  $\alpha_e = e^2/(4\pi)$  is the fine structure constant,  $\zeta(s, \bar{h})$  is the Hurwitz zeta function, and  $\zeta'(-1, \bar{h}) = d\zeta(s, \bar{h})/ds|_{s=-1}$ . For the case with  $G \neq 0$ , the correction was obtained up to the order of  $G^4$ .<sup>60</sup> Recently, we have expressed the correction as a power series in a small parameter  $(g/h)$ .<sup>30</sup>

In finding the light modes in a pure magnetic field, one may be tempted to use the Lagrangian for a pure magnetic field,  $\mathcal{L}_{\text{HES}}^{G=0}(\bar{h})|_{h=B_0}$ . However, the Lagrangian is not sufficient, and a correction of  $O(G)$  [equivalently  $O(g/h)$ ] should be included because the combined fields of the background field and the light field can have  $G \neq 0$  in general ( $\delta \mathbf{E} \cdot \mathbf{B}_0$  is not necessarily zero). The correction is given by

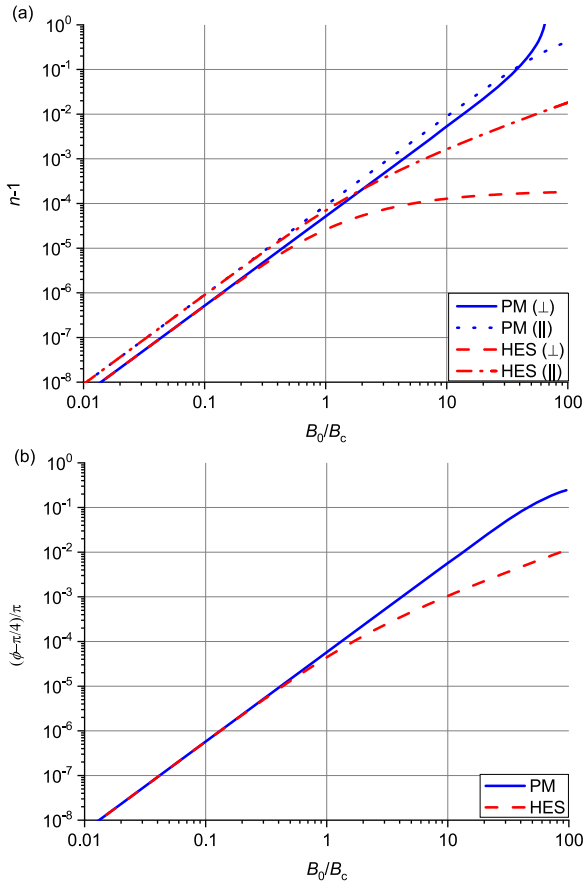
$$\mathcal{L}_{\text{cor}}(\bar{h}, \bar{g}) = \frac{m^4}{8\pi^2 \bar{h}^2} \frac{1}{\bar{g}^2} \left( -\frac{1}{24\bar{h}} + \frac{\ln \bar{h} - \psi^{(0)}(\bar{h})}{12} \right), \quad (54)$$

where  $\bar{g} = m^2/(2eg)$ , and  $\psi^{(0)}(\bar{h})$  is the zeroth-order polygamma function.<sup>30</sup> The coefficients  $\{a^2, ab, b^2, c^2, d^2\}$  are then obtained from (24) by applying the conditions  $h = B_0$  and  $g = 0$  (a purely magnetic background field):

$$\begin{aligned} a^2 &= \frac{16\alpha_e^2 \bar{h}^2}{m^4} \left\{ 2\bar{h}^2 (\psi^{(0)}(\bar{h}) - 1) + \bar{h} \left[ 1 - \ln \frac{\bar{h} \Gamma(\bar{h})^2}{2\pi} \right] + \frac{1}{3} \right\}, \\ ab &= b^2 = 0, \\ c^2 &= -\frac{8\alpha_e^2 \bar{h}^2}{3m^4} \left[ 6\bar{h}^2 - 6\bar{h} \ln \frac{2\pi \bar{h}}{\Gamma(\bar{h})^2} \right] \\ &\quad - \frac{8\alpha_e^2 \bar{h}^2}{3m^4} \left[ 1 + 2\psi^{(0)}(\bar{h}) - 24\zeta'(-1, \bar{h}) + \frac{1}{\bar{h}} \right], \\ d^2 &= \frac{\alpha_e}{6\pi} \left[ 6\bar{h}^2 - 6\bar{h} \ln \frac{2\pi \bar{h}}{\Gamma(\bar{h})^2} - 24\zeta'(-1, \bar{h}) + 2 \ln \bar{h} + 1 \right] + 1. \end{aligned} \quad (55)$$

With  $\{a^2, ab, b^2, c^2, d^2\}$  and  $\{\mathbf{B}_0, \hat{\mathbf{n}}\}$ , we find the light modes in the QED vacuum in an arbitrarily strong magnetic field, as shown in Table III, by following the procedure in Table II and Sec. IV C. The simple expressions for the modes in Table III were obtained by Melrose<sup>61</sup> and Denisov *et al.*,<sup>62</sup> although they gave the coefficients  $\mu$  and  $\epsilon$  as complicated integrals instead of explicit functions as in (55).

As an illustrative example, we compare the light propagation modes of the HES and PM Lagrangians under the assumption that the former approaches the latter in the weak-field limit. In Fig. 2, the departures of the refractive indices and the angle of the parallel polarization vector ( $\phi$  in Fig. 1) from their free-vacuum values are shown as functions of the magnetic field strength, and thus nonzero values represent the effect of the background magnetic field. As  $B$  increases over about  $0.5B_c$ , the HES and PM modes begin to separate. At higher values of  $B$ , the background field effects for HES show a tendency of saturation compared with those for PM. The refractive index  $n_{\perp}$  for PM diverges near  $B = 98.4B_c$  because the denominator in the expression of  $n_{\perp}$  vanishes. These results imply that in dealing with the light propagation around highly magnetized neutron stars (in particular, magnetars), the popularly used PM Lagrangian should



**FIG. 2.** Departures of the refractive indices  $n$  and the parallel polarization vector's angle  $\phi$  from their free-space values as functions of the magnetic field strength for PM and HES Lagrangians: (a)  $n_{\perp} - 1$  and  $n_{\parallel} - 1$ ; (b)  $(\phi - \pi/4)/\pi$ . Here, the parameters  $\eta_1$  and  $\eta_2$  of the PM Lagrangian are chosen to match the HES Lagrangian in the weak-field limit:  $\eta_1/4 = \eta_2/7 = e^4/(360\pi^2 m^4)$ .  $B_c = m^2 c^2 / e\hbar = 4.4 \times 10^{13}$  G. The propagation vector's angle ( $\theta$  in Fig. 1) is  $\pi/4$ .

be replaced with the HES Lagrangian, because the magnetic field strength can surpass the critical field by two orders of magnitude in the vicinity of the magnetar's surface. Furthermore, in such an environment, there is also a significant electric field component along the magnetic field. This electric field will promote the angle of the parallel polarization vector ( $\phi$ ) to become a sensitive measure of the background field effect, in contrast to the electric field-free case.<sup>30</sup> All these increasingly complicated situations can be analyzed by the 3+1 formulation in terms of the concepts familiar in conventional optics.

## VI. CONCLUSION

In this paper, we have provided a general formulation of the light modes in NED, which are described by Plebanski-type Lagrangians, i.e.,  $\mathcal{L} = \mathcal{L}(F, G)$ . Instead of Lorentz-covariant quantities, we have used spatial vectors and dyadics (3+1 formulation)

so that the formulation can be conveniently applied to laboratory experiments or astrophysical observations. We have found that NED can be classified into two cases, nondegenerate and degenerate, of which the latter always possesses a mode with  $n = 1$ . For each case, we have derived general explicit formulas for refractive indices and polarization vectors for a given light propagation direction. As illustrations and practical applications, we have applied the formulation to a magnetic background field described by well-known NED Lagrangians, namely, the post-Maxwellian, Born-Infeld, ModMax, and Heisenberg-Euler-Schwinger QED Lagrangians. We have also advanced the possibility of comparing different Lagrangians in the weak-magnetic-field limit to test the underlying NED Lagrangians in extreme environments such as highly magnetized neutron stars and black holes. Finally, we have provided a streamlined procedure for determining light modes in practical applications (see Table II and Sec. IV C).

By describing NED in the framework of conventional optics, the formulation proposed in this paper can be useful and more accessible for investigating light propagation in NED. The effects of NED, such as modifications of the vacuum, may appear in various phenomena, of which vacuum birefringence is most promise as a target for observation, given the availability of high-precision optical techniques. With terrestrial experiments employing strong permanent magnets<sup>63</sup> or ultra-intense lasers/X-ray free-electron lasers,<sup>58,64</sup> it should be possible to demonstrate vacuum birefringence. Although the strengths of the background fields achievable in current or near-future experiments are far below the critical field, the configuration can be designed as simple and convenient as possible for optimal observations.

On the other hand, ongoing or proposed astrophysical observations of X-rays from highly magnetized neutron stars by for example, the Imaging X-ray Polarimetry Explorer (IXPE),<sup>65</sup> the enhanced X-ray Timing and Polarimetry (eXTP) mission,<sup>66</sup> and the Compton Telescope project,<sup>67</sup> will probe the regime of supercritical magnetic fields,<sup>68,69</sup> although the configuration of background electromagnetic field in this case is complicated and thus needs to be accurately modeled. As the combined electric and magnetic fields in, for instance, the Goldreich-Julian dipole model, vary over macroscopic lengthscales, our 3+1 formulation can provide the local values of the refractive indices and polarization vectors. Then, from this local information, we should be able to find the propagation path of light by solving the transport equations of rays and polarization vectors incorporating birefringence. The formulation presented will provide an essential element and concept for realistic model calculations. However, a realistic model for highly magnetized neutron stars should also include plasma<sup>70</sup> and gravitational effects, which go beyond the scope of the present work and will be addressed in the future.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**Chul Min Kim:** Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Sang Pyo Kim:** Conceptualization (equal); Data curation (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

### DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## REFERENCES

- <sup>1</sup>P. Jordan, J. Ehlers, and R. K. Sachs, “Republication of: Contributions to the theory of pure gravitational radiation. Exact solutions of the field equations of the general theory of relativity II,” *Gen. Relativ. Gravit.* **45**, 2691–2753 (2013).
- <sup>2</sup>K. Akiyama, A. Alberdi, W. Alef, K. Asada, R. Azulay *et al.*, “First M87 event horizon telescope results. IV. Imaging the central supermassive black hole,” *Astrophys. J., Lett.* **875**, L4 (2019).
- <sup>3</sup>K. Akiyama, K. Akiyama, A. Alberdi, W. Alef, J. C. Algaba *et al.*, “First Sagittarius A\* event horizon telescope results. VII. Polarization of the ring,” *Astrophys. J., Lett.* **964**, L25 (2024).
- <sup>4</sup>S. Vagnozzi, R. Roy, Y.-D. Tsai, L. Visinelli, M. Afrin *et al.*, “Horizon-scale tests of gravity theories and fundamental physics from the event horizon telescope image of Sagittarius A\*,” *Classical Quant. Grav.* **40**, 165007 (2023).
- <sup>5</sup>M. Born and L. Infeld, “Foundations of the new field theory,” *Proc. R. Soc. London, Ser. A* **144**, 425–451 (1934).
- <sup>6</sup>G. W. Gibbons and C. A. R. Herdeiro, “Born-infeld theory and stringy causality,” *Phys. Rev. D* **63**, 064006 (2001).
- <sup>7</sup>W. Heisenberg and H. Euler, “Folgerungen aus der Diracschen Theorie des Positrons,” *Z. Phys.* **98**, 714 (1936).
- <sup>8</sup>J. Schwinger, “On gauge invariance and vacuum polarization,” *Phys. Rev.* **82**, 664–679 (1951).
- <sup>9</sup>W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics* (Springer, 1985), pp. 1–241.
- <sup>10</sup>E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer, 1991), pp. 1–250.
- <sup>11</sup>F. Sauter, “Über das Verhalten eines Elektrons im homogenen elektrischen Feld nach der relativistischen Theorie Diracs,” *Z. Phys.* **69**, 742–764 (1931).
- <sup>12</sup>S. P. Kim, H. K. Lee, and Y. Yoon, “Effective action of QED in electric field backgrounds,” *Phys. Rev. D* **78**, 105013 (2008).
- <sup>13</sup>S. P. Kim, H. K. Lee, and Y. Yoon, “Effective action of QED in electric field backgrounds. II. Spatially localized fields,” *Phys. Rev. D* **82**, 025015 (2010).
- <sup>14</sup>D. Strickland and G. Mourou, “Compression of amplified chirped optical pulses,” *Opt. Commun.* **55**, 447–449 (1985).
- <sup>15</sup>C. N. Danson, C. Haefner, J. Bromage, T. Butcher, J.-C. F. Chanteloup *et al.*, “Petawatt and exawatt class lasers worldwide,” *High Power Laser Sci. Eng.* **7**, e54 (2019).
- <sup>16</sup>V. M. Kaspi and A. M. Beloborodov, “Magnetars,” *Annu. Rev. Astron. Astrophys.* **55**, 261–301 (2017).
- <sup>17</sup>N. A. Spaldin and M. Fiebig, “The renaissance of magnetoelectric multiferroics,” *Science* **309**, 391–392 (2005).
- <sup>18</sup>W. Eerenstein, N. D. Mathur, and J. F. Scott, “Multiferroic and magnetoelectric materials,” *Nature* **442**, 759–765 (2006).
- <sup>19</sup>N. A. Spaldin and R. Ramesh, “Advances in magnetoelectric multiferroics,” *Nat. Mater.* **18**, 203–212 (2019).
- <sup>20</sup>Z. Bialynicka-Birula and I. Bialynicki-Birula, “Nonlinear effects in quantum electrodynamics. Photon propagation and photon splitting in an external field,” *Phys. Rev. D* **2**, 2341–2345 (1970).
- <sup>21</sup>S. L. Adler, “Photon splitting and photon dispersion in a strong magnetic field,” *Ann. Phys.* **67**, 599–647 (1971).
- <sup>22</sup>J. Plebański, *Lectures on Non-linear Electrodynamics* (Nordita, 1970), pp. 1–147.
- <sup>23</sup>W. Dittrich and H. Gies, “Light propagation in nontrivial QED vacua,” *Phys. Rev. D* **58**, 025004 (1998).
- <sup>24</sup>V. D. Lorenci, R. Klippert, M. Novello, and J. Salim, “Light propagation in non-linear electrodynamics,” *Phys. Lett. B* **482**, 134–140 (2000).
- <sup>25</sup>F. Karbstein and R. Shaisultanov, “Photon propagation in slowly varying inhomogeneous electromagnetic fields,” *Phys. Rev. D* **91**, 085027 (2015).
- <sup>26</sup>Y. N. Obukhov and G. F. Rubilar, “Fresnel analysis of wave propagation in nonlinear electrodynamics,” *Phys. Rev. D* **66**, 024042 (2002).
- <sup>27</sup>G. F. Rubilar, “Linear pre-metric electrodynamics and deduction of the light cone,” *Ann. Phys.* **514**, 717–782 (2002).
- <sup>28</sup>W.-T. Ni, H.-H. Mei, and S.-J. Wu, “Foundations of classical electrodynamics, equivalence principle and cosmic interactions: A short exposition and an update,” *Mod. Phys. Lett. A* **28**, 1340013 (2013).
- <sup>29</sup>C. M. Kim and S. P. Kim, “Magnetars as laboratories for strong field QED,” *AIP Conf. Proc.* **2874**, 020013 (2024).
- <sup>30</sup>C. M. Kim and S. P. Kim, “Vacuum birefringence at one-loop in a supercritical magnetic field superposed with a weak electric field and application to pulsar magnetosphere,” *Eur. Phys. J. C* **83**, 104 (2023).
- <sup>31</sup>J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, 1999), pp. 555–556.
- <sup>32</sup>S. S. Komissarov, “Electrodynamics of black hole magnetospheres,” *Mon. Not. Roy. Astron. Soc.* **350**, 427–448 (2004).
- <sup>33</sup>A. Kandus, K. E. Kunze, and C. G. Tsagas, “Primordial magnetogenesis,” *Phys. Rep.* **505**, 1–58 (2011).
- <sup>34</sup>S. E. Gralla and T. Jacobson, “Spacetime approach to force-free magnetospheres,” *Mon. Not. Roy. Astron. Soc.* **445**, 2500–2534 (2014).
- <sup>35</sup>L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Elsevier, 1984), pp. 1–480.
- <sup>36</sup>The probe light’s frequency  $\omega$  is assumed to be much smaller than the medium’s characteristic frequency. For the QED vacuum, the characteristic frequency is associated with the electron’s rest mass energy  $m \simeq 0.5$  MeV.
- <sup>37</sup>J. McKenna and P. M. Platzman, “Nonlinear interaction of light in a vacuum,” *Phys. Rev.* **129**, 2354–2360 (1963).
- <sup>38</sup>M. Born, E. Wolf, A. B. Bhatia, P. C. Clemmow, D. Gabor *et al.*, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 7th ed. (Cambridge University Press, 1999), pp. 790–795.
- <sup>39</sup>D. B. Melrose and R. C. McPhedran, *Electromagnetic Processes in Dispersive Media* (Cambridge University Press, 1991), pp. 68–71.
- <sup>40</sup>The cross product of two vectors can be rewritten as a matrix–vector product. For  $\mathbf{n} = (n_1, n_2, n_3)$ , we construct an associated matrix  $\tilde{\mathbf{n}}$  with  $\tilde{n}_{ij} = -\epsilon_{ijk}n_k$ . Then, the following identities hold:  $\mathbf{n} \times \mathbf{A} = \tilde{\mathbf{n}} \cdot \mathbf{A}$ ,  $(\mathbf{n} \times \mathbf{A})\mathbf{B} = (\tilde{\mathbf{n}} \cdot \mathbf{A})\mathbf{B}$ , and  $(\mathbf{A}\mathbf{B}) \cdot \tilde{\mathbf{n}} = -\mathbf{A}(\tilde{\mathbf{n}} \cdot \mathbf{B})$ .
- <sup>41</sup>M. Fiebig, “Revival of the magnetoelectric effect,” *J. Phys. D Appl. Phys.* **38**, R123–R152 (2005).
- <sup>42</sup>R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. (Cambridge University Press, 2013).
- <sup>43</sup>J. S. Heyl and L. Hernquist, “Birefringence and dichroism of the qed vacuum,” *J. Phys. A: Math. Gen.* **30**, 6485–6492 (1997).

- <sup>44</sup>F. Karbstein, “Probing vacuum polarization effects with high-intensity lasers,” *Particles* **3**, 39–61 (2020).
- <sup>45</sup>V. B. Berestetskii, L. P. Pitaevskii, and E. M. Lifshitz, *Quantum Electrodynamics*, 2nd ed. (Butterworth-Heinemann, 1982), Vol. 4, pp. 573–585.
- <sup>46</sup>S. Robertson, “Optical Kerr effect in vacuum,” *Phys. Rev. A* **100**, 063831 (2019).
- <sup>47</sup>J. W. Gibbs and E. B. Wilson, *Vector Analysis: A Text-Book for the Use of Students of Mathematics and Physics* (Yale University Press, 1913), pp. 295–297.
- <sup>48</sup>For a direction vector  $\hat{n}$  and an arbitrary vector  $\mathbf{A}$ , we use the following notation:  $A_n := \mathbf{A} \cdot \hat{n}$ ,  $\mathbf{A}_\perp := \mathbf{A} \times \hat{n}$ ,  $\mathbf{A}_\parallel := (\hat{n} \times \mathbf{A}) \times \hat{n}$ ,  $A^2 := \mathbf{A} \cdot \mathbf{A}$ , and  $A_\perp^2 := A^2 - A_n^2$ .
- <sup>49</sup>V. A. De Lorenci, R. Klippert, S.-Y. Li, and J. P. Pereira, “Multirefringence phenomena in nonlinear electrodynamics,” *Phys. Rev. D* **88**, 065015 (2013).
- <sup>50</sup>T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein’s Equations on the Computer* (Cambridge University Press, 2010), pp. 46–86.
- <sup>51</sup>I. Bandos, K. Lechner, D. Sorokin, and P. K. Townsend, “Nonlinear duality-invariant conformal extension of Maxwell’s equations,” *Phys. Rev. D* **102**, 121703 (2020).
- <sup>52</sup>D. P. Sorokin, “Introductory notes on non-linear electrodynamics and its applications,” *Fortschr. Phys.* **70**, 2200092 (2022).
- <sup>53</sup>W. Dittrich, “One-loop effective potentials in quantum electrodynamics,” *J. Phys. A: Math. Gen.* **9**, 1171–1179 (1976).
- <sup>54</sup>K. A. Bronnikov, “Regular black holes sourced by nonlinear electrodynamics,” in *Regular Black Holes: Towards a New Paradigm of Gravitational Collapse*, edited by C. Bambi (Springer, 2023), pp. 37–67.
- <sup>55</sup>J. Maldacena, “Comments on magnetic black holes,” *J. High Energy Phys.* **2021**, 79.
- <sup>56</sup>V. F. Weisskopf, “Ueber die elektrodynamik des vakuums auf grund des quanten-theorie des elektrons,” *Dan. Mat. Fys. Medd.* **14**, 1–39 (1936).
- <sup>57</sup>B. King and T. Heinzl, “Measuring vacuum polarization with high-power lasers,” *High Power Laser Sci. Eng.* **4**, e5 (2016).
- <sup>58</sup>Q. Yu, D. Xu, B. Shen, T. E. Cowan, and H.-P. Schlenvoigt, “X-ray polarimetry and its application to strong-field quantum electrodynamics,” *High Power Laser Sci. Eng.* **11**, e71 (2023).
- <sup>59</sup>S. P. Kim and H. K. Lee, “Quantum electrodynamics actions in supercritical fields,” *J. Korean Phys. Soc.* **74**, 930–934 (2019).
- <sup>60</sup>J. S. Heyl and L. Hernquist, “Analytic form for the effective Lagrangian of QED and its application to pair production and photon splitting,” *Phys. Rev. D* **55**, 2449–2454 (1997).
- <sup>61</sup>D. B. Melrose, *Quantum Plasmadynamics: Magnetized Plasmas* (Springer, 2013), pp. 372–378.
- <sup>62</sup>V. I. Denisov, E. E. Dolgaya, and V. A. Sokolov, “Nonperturbative QED vacuum birefringence,” *J. High Energy Phys.* **2017**, 105.
- <sup>63</sup>A. Ejlli, F. Della Valle, U. Gastaldi, G. Messineo, R. Pengo *et al.*, “The PVLAS experiment: A 25 year effort to measure vacuum magnetic birefringence,” *Phys. Rep.* **871**, 1–74 (2020).
- <sup>64</sup>F. Karbstein, C. Sundqvist, K. S. Schulze, I. Uschmann, H. Gies *et al.*, “Vacuum birefringence at X-ray free-electron lasers,” *New J. Phys.* **23**, 095001 (2021).
- <sup>65</sup>R. Taverna *et al.*, “Polarized X-rays from a magnetar,” *Science* **378**, 646–650 (2022).
- <sup>66</sup>A. Santangelo, S. Zane, H. Feng, R. Xu, V. Doroshenko *et al.*, “Physics and astrophysics of strong magnetic field systems with eXTP,” *Sci. China: Phys., Mech. Astron.* **62**, 29505 (2019).
- <sup>67</sup>Z. Wadiasingh, G. Younes, M. G. Baring, A. K. Harding, P. L. Gonthier *et al.*, “Magnetars as astrophysical laboratories of extreme quantum electrodynamics: The case for a compton telescope,” *Bull. Am. Astron. Soc.* **51**, 292 (2019).
- <sup>68</sup>S. A. Olausen and V. M. Kaspi, “The McGill magnetar catalog,” *ApJS* **212**, 6 (2014).
- <sup>69</sup>I. Caiazzo, D. González-Caniulef, J. Heyl, and R. Fernández, “Probing magnetar emission mechanisms with X-ray spectropolarimetry,” *Mon. Not. R. Astron. Soc.* **514**, 5024–5034 (2022).
- <sup>70</sup>C. Wang and D. Lai, “Wave modes in the magnetospheres of pulsars and magnetars,” *Mon. Not. Roy. Astron. Soc.* **377**, 1095–1112 (2007).