Decomposition of regular hypergraphs

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Abstract

A *d-block* is a $0,1$-matrix in which every row has sum d. Let S_n be the set of pairs (k,l) such that the columns of any $(k+l)$-block with n rows split into a k-block and an l-block. For $n \ge 5$, we prove the general necessary condition that $(k,l) \in S_n$ only if each element of $\{1,\ldots,n\}$ divides k or l. We also determine S_n for $n \le 5$. Trivially, $S_1 = S_2 = \mathbb{N}\times\mathbb{N}$. Also $S_3 = \{(k,l)\colon\ 2 \mid kl\}$, $S_4 = \{(k,l)\colon\ 6 \mid kl \text{ and } \min\{k,l\} > 1\}$, and $S_5 = \{(k,l)\colon\ 3,4,5 \text{ each divide } k \text{ or } l, \text{ plus } 11 \ne \min\{k,l\} > 7\}$.

1 Introduction

Our problem is most simply expressed in the language of $0,1$-matrices. A *block* is a $0,1$-matrix M whose rows all have the same sum; we denote the common sum by $\sigma(M)$. We use *d-block* to mean a block M with $\sigma(M) = d$. Given a $(k+l)$-block with $k,l \in \mathbb{N}$ (where $\mathbb{N}$ is the set of positive integers), a *(k,l)-split* is a partition of the columns into two sets such that the resulting submatrices are a k-block and an l-block. A d-block M is *indecomposable* if for all (k,l) with $k+l=d$, there is no (k,l)-split of M.

Trivially, every $(k+l)$-block with one row has a (k,l)-split. This also holds for two rows, since columns of the forms $(0,1)^T$ and $(1,0)^T$ are equinumerous and can be paired. For $n \in \mathbb{N}$, let S_n be the set of pairs (k,l) such that every $(k+l)$-block with n rows admits a (k,l)-split. Adding a row imposes additional restrictions, so $S_{n+1} \subseteq S_n$ for all n. We have noted $S_1 = S_2 = \mathbb{N}\times\mathbb{N}$. In this paper, we determine S_3, S_4, and S_5, and we prove a general necessary condition for $n \ge 5$.

Splitting of d-blocks into blocks with smaller row-sums has been studied in the language of hypergraphs. Each edge of a hypergraph is a subset of the vertex set, and distinct edges may have the same vertex set. A hypergraph is *d-regular* if every vertex lies in exactly d edges. The *incidence matrix* of a hypergraph is the 0,1-matrix with rows indexed by

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the vertices and columns indexed by the edges such that entry (v, e) is 1 if and only if vertex v row belongs to edge e. Thus a hypergraph is d-regular if and only if its incidence matrix is a d-block. A regular hypergraph H is *indecomposable* if has no nontrivial regular spanning proper subhypergraph, which is just the statement that its incidence matrix is an indecomposable d-block.

Motivated by questions in game theory (see the survey [4]), researchers studied the maximum possible degree of indecomposable regular hypergraphs with n vertices. That is, the value $D(n)$ is the maximum d such that some d-block with n rows is indecomposable. Huckemann and Jurkat (see [4]) proved that $D(n)$ is finite for all n (reproved in another way by Alon and Berman [1]), and with Shapley they proved $D(n) \le (n+1)^{(n+1)/2}$ (again see [4]). As a lower bound, Shapley proved $D(n) > 2^{n-1}/(n-1)$ for $n > 2$, improved to $D(n) > 2^{n-3}$ for $n > 2$ by van Lint and Pollak. Alon and Vũ [2] later proved the asymptotic formula $D(n) = n^{(1+o(1))n/2}$ (they showed in fact that this formula solves three problems). Their formula is close to the upper bound of Huckemann, Jurkat, and Shapley. Füredi [3] considered the restriction of the problem to hypergraphs in which every edge has size t. Kézdy, Lehel, and Powers [5] gave an application of the bounds on $D(n)$ to a problem involving weighted hypergraphs and the selection of a "consensus" vertex.

As far as we know, the exact values of $D(n)$ are known only for $1 \le n \le 5$ [4]; they are $1, 1, 2, 3, 5$, respectively. We will use these values to study S_n. Recall that $S_n = \mathbb{N} \times \mathbb{N}$ when $n \le 2$. For $n = 3$, a bit more thought yields $S_3 = \{(k, l)\colon\ 2 \mid kl\}$. We also prove $S_4 = \{(k, l)\colon\ 6 \mid kl \text{ and } \min\{k, l\} > 1\}$ and $S_5 = \{(k, l)\colon\ 3, 4, 5 \text{ each divide } k \text{ or } l, \text{ and } 11 \ne \min\{k, l\} > 7\}$. Note that the condition for S_4 implies that $(1, l) \notin S_n$ for $l \in \mathbb{N}$ and $n \ge 4$. Thus there is no nontrivial (d, n) such that every d-regular hypergraph with n vertices has a perfect matching.

The divisibility requirement for $n = 5$ is the special case for $n = 5$ of a general necessary condition for membership in S_n, which we develop in Section 4:

Theorem 1.1. *For $n \ge 5$, if $(k, l) \in S_n$, then each element of $\{1, \dots, n\}$ divides k or l.*

For $2 \le n \le 4$, the condition is not quite necessary; changing $\{1, \dots, n\}$ to $\{1, \dots, n-1\}$ yields a weaker condition that characterizes S_n in those cases. Since the result of van Lint and Pollak cited above yields $D(n) > n$ when $n \ge 6$, while $D(n) = n - 1$ for $2 \le n \le 4$, we pose the following conjecture.

Conjecture 1.2. *A necessary condition for $(k, l) \in S_n$ is that each element of $\{1, \dots, D(n)\}$ divides k or l. If $\min\{k, l\}$ is sufficiently large, then this condition is also sufficient.*

Our results for $n \le 5$ agree with Conjecture 1.2. Since $D(n) \ge n$ for $n \ge 6$, Conjecture 1.2 strengthens Theorem 1.1.

We use the known values of $D(n)$ in proving both necessity and sufficiency of the description of S_n for $n \le 5$. Section 2 characterizes S_3 and outlines our general approach to determining S_n; the details for $n \in \{4,5\}$ follow in later sections. The method could perhaps settle Conjecture 1.2 for more values of n once the corresponding values of $D(n)$ are known.

Although we use $D(n)$ to determine S_n, it is worth noting that $(k,l) \in S_n$ implies $D(n) < k+l$. Hence $m(n) > D(n)$, where $m(n) = \min\{k+l\colon (k,l) \in S_n\}$. A referee pointed out that, due to the divisibility requirement, Conjecture 1.2 would yield a much larger threshold: $m(n) \geq e^{D(n)/2+o(D(n))}$.

2 General approach

To illustrate our method, we first characterize S_3. An equivalent statement was proved by André Kündgen (unpublished). When A and B are matrices with the same number of rows, let $A\!:\!B$ denote their concatenation, taking the union of the column sets as multisets. We use mA to denote the concatenation of m copies of A. Also, when B is a submatrix of A consisting of full columns, let $A \setminus B$ denote the matrix obtained by deleting those columns. We say that B is a *block in M* when B is a block consisting of full columns of M.

Theorem 2.1. $S_3 = \{(k,l)\colon\ 2 \mid kl\}$

Proof. Let M_1 and M_2 be the 1-block and 2-block with three rows shown below. Note that M_2 is indecomposable; it contains no 1-block.

$$M_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad M_2 = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}$$

Necessity. Let $M = \frac{k+l}{2}M_2$; this is a $(k+l)$-block. We claim that every block in M has even row-sum. If a block in M with odd row-sum contains a copy of each column of M_2, then deleting these three columns yields a smaller block with odd row-sum. Hence a minimal block B among those with odd row-sum uses only copies of at most two columns of M_2. Now some row has no 0, while another row does have a 0, so B is not a block. Thus if kl is odd, then M is a $(k+l)$-block with no (k,l)-split; this yields $(k,l) \notin S_3$.

Sufficiency. By symmetry, we may assume $2 \mid k$. Since $k \geq 2$ and $l \geq 1$, we have $d \geq 3$, where $d = k+l$. Since $D(3) = 2$, every d-block M with $d \geq 3$ (and three rows) decomposes into blocks with row-sum at most 2. We therefore obtain a 2-block A in M (possibly the concatenation of two 1-blocks). When $k = 2$, this completes the decomposition. To complete a proof by induction on k, when $k > 2$ we combine A with a $(k-2)$-block from the $(k-2,l)$-split that the induction hypothesis guarantees for $M \setminus A$. ☐

The sufficiency proof that the pairs not excluded from S_n actually do belong to S_n uses the value of $D(n)$ and induction on $k+l$. For the base case, we will need to check that when

(k,l) is in the specified set and $k+l \le D(n)$, every $(k+l)$-block has a (k,l)-split. When $n \le 5$, we have $D(n) \le n$, and there is not much to check in the base case.

For the induction step, when $k+l > D(n)$, every $(k+l)$-block M decomposes into blocks with row-sum at most $D(n)$. If we can always find a block B in M with $\sigma(B) \le D(n)$ such that reducing k or l by $\sigma(B)$ yields another pair (k', l') in the specified set, then combining B with one of the blocks in a (k', l')-split of $M \setminus B$ guaranteed by the induction hypothesis completes the proof.

For the necessity of the characterization, our proofs that exclude a pair (k,l) from S_n are implementations of the next lemma, which we implicitly used in proving Theorem 2.1. Let $[m]$ denote the set $\{1, \ldots, m\}$, and let M_0 denote a matrix with no columns, so $M : M_0 = M$.

Definition 2.2. A positive integer q is *n-robust* if for all r with $0 \le r < q$, there exist an indecomposable q-block M_q and an indecomposable r-block M_r (both with n rows) such that for all $p \in \mathbb{N}$ the row-sum of any block in $pM_q : M_r$ is congruent to 0 or r modulo q.

Lemma 2.3. *If q is n-robust and $(k,l) \in S_n$, then q divides k or l.*

Proof. Suppose that q is n-robust and does not divide k or l. Let $s = \lfloor k/q \rfloor$ and $i = k - sq$, and let $t = \lfloor l/q \rfloor$ and $j = l - tq$. Choose $r \in [q]$ such that $r \equiv i + j \mod q$. Given the resulting M_q and M_r guaranteed by the definition of n-robust, let $M = (s+t)M_q : M_r$ or $M = (s+t+1)M_q : M_r$, depending on whether $i + j \le q$ or not. Now M is a $(k+l)$-block. By the definition of n-robust, M does not contain a k-block, so $(k,l) \notin S_n$. □

Note that if q is n-robust, then $q \le D(n)$; this motivates Conjecture 1.2. The difficulty in applying Lemma 2.3 is finding the needed q-block and r-block (for each r) and checking that the concatenations do not contain blocks with undesirable row-sums. The lemma does not save any work; it only states the plan. If we supply the specified indecomposable q-block and r-block for each prime power q up to $D(n)$ and each $r \in [q]$, then a necessary condition for $(k,l) \in S_n$ will be that each prime power up to $D(n)$ divides k or l.

It is not known whether there is an indecomposable d-block with n rows (indecomposable d-regular n-vertex hypergraph) whenever $d < D(n)$. Nevertheless, our characterization of S_n for $n \le 5$ includes exhibiting such blocks for $n \le 5$.

Recall that $S_{n+1} \subseteq S_n$. Since $(k,l) \in S_3$ requires kl to be even, it therefore follows for $n \ge 3$ that kl must be even when $(k,l) \in S_n$. For larger n we can also eliminate the pairs containing a 1.

Definition 2.4. Let M_1, M_2, and M_3 henceforth denote the matrices below.

$$M_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Lemma 2.5. *If $n \geq 4$, then no pair (k, l) with $k = 1$ belongs to S_n.*

Proof. Since $S_{n+1} \subseteq S_n$, it suffices to prove this for $n = 4$. If $(k, l) \in S_4$, then kl must be even. Thus it suffices to provide, for each even l, an $(l + 1)$-block containing no 1-block. With M_1, M_2, M_3 defined as above, let $M = (l/2 - 1)M_2 : M_3$. Since M_2 is a 2-block and M_3 is a 3-block, M is an $(l+1)$-block. Since no column of M equals M_1 and the sum of any two columns has at least one 2, M contains no 1-block. □

3 Matrices with four rows

When $n = 4$, the matrices M_1, M_2, M_3 will play the roles of the matrices needed to apply the necessary condition in Lemma 2.3. When we speak of a matrix having a particular "form", we are allowing permutations of the columns.

Lemma 3.1. *For $p \in \mathbb{N}$ and $0 \leq r < q \leq 3$, every block in $pM_q : M_r$ has the form $p'M_q$ or $p'M_q : M_r$ for some p'. Consequently, if $(k, l) \in S_n$ for $n \geq 4$, then 3 divides k or l.*

Proof. The second statement follows from the first by Lemma 2.3 and $S_{n+1} \subseteq S_n$, since every block of the specified form has row-sum congruent to 0 or r modulo q. For $q \leq 2$ the first statement is trivial, so assume $q = 3$.

Let B be a smallest block in $pM_3 : M_r$ not having the desired form. If B contains a copy of each column in M_3, then there is a smaller such block. Hence we may assume that some column of M_3 is not use in B.

For $r = 0$, now B has a row with no 0 and a row with 0 and is not a block.

For $r = 1$, by the previous case we may assume that B contains the column M_1. A block B containing M_1 consists of that column and a (possibly empty) block from pM_3. Hence B has the specified form.

For $r = 2$, the block B must use the one column in M_2 not in M_3. If B contains copies of both other columns of M_2, then the case $r = 0$ applies. Otherwise, B has a row having no 0 and a row that has a 0 and cannot be a block. □

Theorem 3.2. $(k, l) \in S_4$ *if and only if* $6 \mid kl$ *and* $\min\{k, l\} \neq 1$.

Proof. By Theorem 2.1 and $S_{n+1} \subseteq S_n$, we have $2 \mid kl$. By Lemma 3.1, we have $3 \mid kl$. By Lemma 2.5, $\min\{k, l\} \neq 1$. Hence the conditions are necessary.

For sufficiency, suppose that $6 \mid kl$. By symmetry, there are two cases: either $k = 3s$ and $l = 2t$ for positive integers s and t, or $k = 6s$ with s a positive integer and $l \geq 2$.

Case 1. *$k = 3s$ and $l = 2t$ for positive integers s and t.* We use induction on $s + t$. Let M be a $(k + l)$-block. Since $D(4) = 3$, every d-block with $d \geq 4$ decomposes into 1-blocks, 2-blocks, and 3-blocks. If these are all 1-blocks, then M contains a k-block.

Thus, we may assume that M contains a 3-block or a 2-block A. If A is a 3-block and $s = 1$, or A is a 2-block and $t = 1$ (covering the base case $(3, 2)$), then A or its complement is the desired k-block. Otherwise, we apply the induction hypothesis to $M \setminus A$ and combine A with a $(k-3)$-block or a $(l-2)$-block in $M \setminus A$ to obtain a k-block or an l-block in M.

Case 2. $k = 6s$ *and* $l \geq 2$. We use induction on l. The cases with $l \in \{2, 3, 4\}$ appear in Case 1 as $(3 \cdot 2s, 2 \cdot 1)$, $(3 \cdot 1, 2 \cdot 3s)$, and $(3 \cdot 2s, 2 \cdot 2)$, respectively. For $l \geq 5$, since we may assume that any $(k+l)$-block contains a 2-block or a 3-block A, we can apply the induction hypothesis using k and $l-2$ or $l-3$ to $M \setminus A$. □

4 A General Necessary Condition

In this section we prove that when $n \geq 5$ and $1 \leq q \leq n$, membership of (k, l) in S_n requires that q divides k or l. This necessary condition is not sufficient. In the next section, we will exclude additional pairs when $n = 5$ to obtain the complete description of S_5.

Definition 4.1. For $1 \leq i \leq n$, we define an i-block $M_i(n)$ with n rows. For $n = 5$ these are listed below; note that $M_3(5), M_4(5), M_5(5)$ have $4, 6, 7$ columns, respectively.

$$\underset{M_1(5)}{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}} \quad \underset{M_2(5)}{\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\\1&1&0\\1&1&0\end{bmatrix}} \quad \underset{M_3(5)}{\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&0&1\\1&1&1&0\end{bmatrix}} \quad \underset{M_4(5)}{\begin{bmatrix}0&0&1&1&1&1\\1&1&0&1&1&0\\1&1&1&0&0&1\\1&1&1&0&1&0\\1&1&1&1&0&0\end{bmatrix}} \quad \underset{M_5(5)}{\begin{bmatrix}0&0&1&1&1&1&1\\1&1&0&0&1&1&1\\1&1&1&1&0&0&1\\1&1&1&1&0&1&0\\1&1&1&1&1&0&0\end{bmatrix}}$$

For $n \geq 6$, let $M_1(n)$ be the all-1 column vector, and define $M_2(n)$ by repeating the bottom row of $M_2(n-1)$. For $3 \leq q \leq n$, obtain $M_q(n)$ from $M_{q-1}(n-1)$ by first appending a 1 to the end of each row and then adding an nth row in which the first q entries are 1 and the last one or two entries are 0 (one 0 when $q \leq n-2$, two when $q \in \{n-1, n\}$). Note that $M_q(n)$ has $q+1$ columns when $2 \leq q \leq n-2$ and $q+2$ columns When when $q \in \{n-1, n\}$. Graphically,

$$M_q(n) = \left[\begin{array}{ccc|c} & M_{q-1}(n-1) & & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \\ \hline 1 & \cdots & 1 & 0 \end{array}\right] \text{ for } 3 \leq q \leq n-2; \qquad M_q(n) = \left[\begin{array}{cccc|c} & M_{q-1}(n-1) & & & \begin{matrix}1\\ \vdots\\ 1\end{matrix} \\ \hline 1 & \cdots & 1 & 0 & 0 \end{array}\right] \text{ for } n-1 \leq q \leq n.$$

Note also that for $n = 5$ the last column in $M_q(n)$ is unique and is not all 1. Therefore, inductively we obtain the same statement for all $n \geq 5$; this property will be important.

Lemma 4.2. *For $n, p, q \in \mathbb{N}$ with $n \geq 5$ and $q \leq n$, every block in $pM_q(n)$ has the form $p'M_q(n)$ for some p' with $p' \leq p$.*

Proof. The statement is trivial for $q = 1$. For $q \in \{2, 3\}$, the statement of the conclusion holds for the matrix M_q of Definition 2.4, by Lemma 3.1. Hence it also holds for $pM_q(n)$, since the matrix $M_q(n)$ arises from M_q by making extra copies of some row.

For $q \geq 4$, we use induction on n. We begin for $n = 5$ by proving the statement for $q \in \{4, 5\}$. First for $q = 5$, let B be a block in $pM_5(5)$. There are five types of columns from $M_5(5)$; from left to right, let a, b, c, d, e denote their multiplicities in B, respectively. Since a block must have the same number of 0s in each row, the five row constraints give $a = b = c+d = c+e = d+e$. These equations require $a = b = 2c = 2d = 2e$, so there is just one parameter. Also, since $a = b = 2c$, we can view the block as having equal multiplicity for each of the seven columns of $M_5(5)$. Hence B has the desired form.

For $q = 4$ and $n = 5$, let B be a block in $pM_4(5)$. Each column of $M_4(5)$ appears in $M_5(5)$ except the last. Let z be the multiplicity in B of the last column of $M_4(5)$, and let a, b, c, d be the multiplicities of the other columns of $M_4(5)$, named as in $M_5(5)$. Since each row of B has the same number of 0s, we obtain $a = b + z = c + d = c + z = d + z$. These equations require $a = 2b = 2c = 2d = 2z$, and we can view the block as having equal multiplicity for each of the six columns of $M_4(5)$. Hence B has the desired form.

For the induction step, consider $n \geq 6$ and $4 \leq q \leq n$. Let B be a block in $pM_q(n)$. Let M' be the matrix consisting of the first $n-1$ rows of $M_q(n)$. Let v be the last column of $M_q(n)$, and let z be the number of copies of v in B. Let B' be the matrix obtained by removing the copies of v from B and deleting the last row. Since the copies of v contributed z to the sum of each row in B other than the last row, B' is a block in pM'.

Since $pM' = pM_{q-1}(n-1)$, the induction hypothesis implies that B' consists of p' copies of each column of $M_{q-1}(n-1)$, for some p'. Thus $\sigma(B') = p'(q-1)$. Since $M_q(n-1)$ has q copies of 1 in the bottom row before v, and each of those columns appears p' times in B, we have $\sigma(B) = p'q$. Therefore, $z = p'$, and B has the desired form. □

Setting $p = 1$ in Lemma 4.2 yields the statement that $M_q(n)$ is indecomposable.

Lemma 4.3. *For $p \in \mathbb{N}$, every block in $pM_q(5) : M_r(5)$ for $0 \leq r < q \leq 5$ has the form $p'M_q(5) : M_r(5)$ or $p'M_q(5)$ for some p' with $p' \leq p$. In particular, q is 5-robust for $1 \leq q \leq 5$.*

Proof. Lemma 4.2 is the case $r = 0$. For $r = 1$, since the one column of $M_1(5)$ is all 1s, Lemma 4.2 again applies.

Consider now $r > 1$. Observe that every column of $M_r(5)$ lies in $M_q(5)$ except the last column of $M_r(5)$. Hence Lemma 4.2 implies that a block B in $pM_q(5) : M_r(5)$ with $\sigma(B)$ not divisible by q must use the one copy of the last column of $M_r(5)$. Let the multiplicities of the other columns again be a, b, c, d, e, using the notation for columns of $M_5(5)$ as in Lemma 4.2. In considering $r \in \{2, 3, 4\}$, let x, y, z respectively be the multiplicity of the last column in

$M_r(5)$. Except for copies of that column, we count all columns used as copies of columns of $M_q(5)$.

We again count the 0s in each row. For each case (q, r), these counts appear below from row 1 to row 5 under "constraints from 0s"; the five values must be equal. The equalities allow us to compute all multiplicities in terms of c as in the next section of the table. The final column then counts the 1s in each row of B. In each case, the row-sum is congruent to r modulo q, and B has the desired form. □

q	r	constraints from 0s					a	b	c	d	e	x	y	z	$\sigma(B)$
3	2	a	b	$c{+}1$	$c{+}1$	$y{+}1$	$c{+}1$	$c{+}1$	c	0	0	1	c	0	$3c{+}2$
4	2	a	$b{+}z$	$c{+}d{+}1$	$c{+}z{+}1$	$d{+}z{+}1$	$2c{+}1$	$c{+}1$	c	c	0	1	0	c	$4c{+}2$
4	3	a	$b{+}z$	$c{+}d$	$c{+}z$	$d{+}z{+}1$	$2c{-}1$	c	c	$c{-}1$	0	0	1	$c{-}1$	$4c{-}1$
5	2	a	b	$c{+}d{+}1$	$c{+}e{+}1$	$d{+}e{+}1$	$2c{+}1$	$2c{+}1$	c	c	c	1	0	0	$5c{+}2$
5	3	a	b	$c{+}d$	$c{+}e$	$d{+}e{+}1$	$2c{-}1$	$2c{-}1$	c	$c{-}1$	$c{-}1$	0	1	0	$5c{-}2$
5	4	a	$b{+}1$	$c{+}d$	$c{+}e{+}1$	$d{+}e{+}1$	$2c$	$2c{-}1$	c	c	$c{-}1$	0	0	1	$5c{-}1$

Lemma 4.4. *For $p, n \in \mathbb{N}$ with $n \ge 5$, every block in $pM_q(5){:}M_r(5)$ for $0 \le r < q \le n$ has the form $p'M_q(n){:}M_r(n)$ or $p'M_q(n)$ for some p' with $p' \le p$. Thus q is n-robust for $q \le n$.*

Proof. Lemma 4.3 is the case $n = 5$; we use that as the basis for induction on n. For larger n, the claim for $r \le 1$ is Lemma 4.2. Consider $r \ge 2$ (and hence $q > 2$), and let B be a block in $pM_q(n){:}M_r(n)$, with $t = \sigma(B)$.

Let $r' = \max\{r-1, 2\}$. Arrange the columns of B by placing the copies of $[1 \cdots 1\ 0]^T$ at the right end. This yields the following form of B.

$$B = \left[\begin{array}{c|c} M & J \\ \hline H & 0 \ \cdots \ 0 \end{array}\right],$$

where J is an all-1 matrix. Let z be the number of columns in J. Since the last columns of $M_q(n)$ and $M_r(n)$ are unique, M is a block in $pM_{q-1}(n-1){:}M_{r'}(n-1)$, with $\sigma(M) = t - z$.

By the induction hypothesis, M consists of the columns of $p'M_{q-1}(n-1) : M_{r'}(n-1)$ or $p'M_{q-1}(n-1)$, for some p' with $0 \le p' \le p$. Thus $t - z$ is $p'(q-1) + r'$ or $p'(q-1)$, respectively.

By the uniqueness of the final columns in $M_q(n)$ and $M_r(n)$, each copy of $M_{q-1}(n-1)$ or $M_{r'}(n-1)$ in M extends by adding the portion of H below it to become the matrix obtained from $M_q(n)$ or $M_r(n)$ by deleting the final column (or that full matrix in the case of $r = 2$). Thus $t \in \{p'q + r, p'q\}$, depending on whether $M_{r'}(n-1)$ appears in M. We conclude that z is $p' + r - r'$ or p', respectively. Hence the final columns provide exactly what is needed to conclude that B has the form $p'M_q(n){:}M_r(n)$ or $p'M_q(n)$. □

Applying Lemma 2.3, we now have the following theorem.

Theorem 4.5. *For $n \geq 5$, if $(k,l) \in S_n$, then $q \in \{1,\dots,n\}$ divides k or l.*

5 Matrices with five rows

In this section, we determine S_5. Lemma 4.3 showed that q is 5-robust for $1 \leq q \leq 5$. Thus Lemma 2.3 implies that for every $(k,l) \in S_5$, each of $\{3,4,5\}$ divides k or l. Also we have forbidden $\min\{k,l\} = 1$. These conditions are not sufficient; the characterization of S_5 excludes additional pairs when $\min\{k,l\}$ is small.

Lemma 5.1. *If $(k,l) \in S_5$, then $3,4,5$ each divide k or l, plus $11 \neq \min\{k,l\} > 7$.*

Proof. By Lemma 4.3, the divisibility condition is necessary. Restricting to $k \geq l$, in this proof we exclude pairs of the form

$$\{(20s,1),(20s,2),(20s,3),(15s,4),(12s,5),(20s,6),(20s,7),(20s,11)\}$$

for each positive integer s. Lemma 4.3 already excludes $(20s,2)$, $(20s,7)$, and $(20s,11)$ when $3 \nmid s$, and Lemma 2.5 excludes $(20s,1)$, but the argument here for the other cases also handles these.

For each case of (k,l), we list below a $(k+l)$-block that we will show has no (k,l)-split. The matrices M_3, M_4, M_5 are as in Definition 4.1. We set $M = (\alpha s - \beta)M_i : \gamma M_j$ to consider $k = i\alpha s$ and $l = \gamma j - \beta i$. We group the cases by the matrix M_j. When $5 \mid k$, we use M_5 as the main repeated block; in the one case where $5 \nmid k$ and $l = 5$, we use M_4.

k	l	$(k+l)$-block M	k	l	$(k+l)$-block M
$20s$	1	$(4s-3)M_5 : 4M_4$	$20s$	2	$(4s-2)M_5 : 4M_3$
$20s$	6	$(4s-2)M_5 : 4M_4$	$20s$	7	$(4s-1)M_5 : 4M_3$
$20s$	11	$(4s-1)M_5 : 4M_4$	$15s$	4	$(3s-1)M_5 : 3M_3$
$20s$	3	$(4s-1)M_5 : 2M_4$	$12s$	5	$(3s-1)M_4 : 3M_3$

In Lemma 4.2, we showed that blocks formed using only columns from pM_q have row-sum divisible by q. Also, for $i < q \leq 5$, each column of M_i except the last appears in M_q, and we showed that a block using one copy of this special column plus columns from M_q has row-sum congruent to i modulo q. In each case l is outside the achievable class.

Now up to γ copies of the exceptional column are available to use in forming an l-block. We use the same technique as before to eliminate these cases; the fact that we only need to exclude row-sum l itself instead of a full congruence class is crucial.

For M_4, the special column is $(1,0,1,0,0)^T$; for M_3, it is $(1,1,1,1,0)^T$. Consider a block B in M using x copies of the special column. Again let a,b,c,d,e, respectively, denote the number of copies of the five columns in M_5 that are used in B. In the last case, let z be the number of copies of the rightmost column of M_4. As before, we first obtain constraints on these multiplicities by ensuring that all rows have the same number of 0s. These determine the multiplicities in terms of c and x, which in turn yields a formula for the row sum. We then argue that l is not achievable. In the computation, several cases combine.

M	constraints from 0s					a	b	c	d	e	$\sigma(B)$
$pM_5 : \gamma M_4$	a	$b+x$	$c+d$	$c+e+x$	$d+e+x$	$2c$	$2c-x$	c	c	$c-x$	$5c-x$
$pM_5 : 4M_3$	a	b	$c+d$	$c+e$	$d+e+x$	$2c-x$	$2c-x$	c	$c-x$	$c-x$	$5c-2x$
$pM_4 : 3M_3$	a	$b+z$	$c+d$	$c+z$	$d+z+x$	$2c-x$	c	c	$c-x$	0	$4c-x$

To form an l-block in $pM_5 : \gamma M_j$, we need $x \le q$. In each case, the requirement that d and e are nonnegative yields $c \ge x$. Also the row-sum in the repeated block M_i is the coefficient on c in $\sigma(B)$. To achieve $\sigma(B) = l$, this fixes the congruence class of x modulo i. Since $x \le \gamma$ and $c \ge x$, in each case this produces too large a value of l.

For $5 \mid k$ and $l \equiv 1 \mod 5$, an l-block in $pM_5 : 4M_4$ requires $x = 4$, but then $l \ge 16$.

For $5 \mid k$ and $l \equiv 2 \mod 5$, an l-block in $pM_5 : 4M_3$ requires $x = 4$, but then $l \ge 12$.

For $5 \mid k$ and $l = 3$, an l-block in $pM_5 : 2M_4$ requires $x = 2$, but then $l \ge 8$.

For $5 \mid k$ and $l = 4$, an l-block in $pM_5 : 3M_3$ requires $x = 3$, but then $l \ge 9$.

For $4 \mid k$ and $l = 5$, an l-block in $pN_4 : 3M_3$ requires $x = 3$, but then $l \ge 9$. □

The final contradictions in the proof of Theorem 5.1 show how delicate these exceptions are. Each case requires l to be at least in the next higher congruence class modulo i. Indeed, after excluding these small values of l, the conditions are sufficient.

Theorem 5.2. *$(k, l) \in S_5$ if and only if $3, 4, 5$ each divide k or l, and also $11 \ne \min\{k, l\} > 7$.*

Proof. Necessity was established in Lemma 5.1. For sufficiency, we consider explicitly the pairs that have not been excluded. We may assume by symmetry that k is divisible by at least two of $\{3, 4, 5\}$. Note that the pairs $(60r, 8)$, $(60r, 9)$, and $(60r, 10)$ have the form $(15s, 4t)$, $(20s, 3t)$, or $(12s, 5t)$, respectively. Hence it suffices to show that for $s \ge 1$ the following pairs lie in S_5:

$$\{(20s, 3t) \colon t \ge 3\} \qquad \{(15s, 4t) \colon t \ge 2\}$$
$$\{(12s, 5t) \colon t \ge 2\} \qquad \{(60s, t) \colon t \ge 12\}.$$

Since $D(5) = 5$, every block M with five rows decomposes into indecomposable blocks with row-sums at most 5. These blocks provide a partition of the integer $\sigma(M)$. Our task is to show that every partition of $k + l$ whose parts are all at most 5 splits into portions summing to k and to l when (k, l) lies in a family listed above. Let b_r be the number of copies of r in the partition. We may assume that no two parts sum to at most 5, because it then suffices to consider the partition obtained by replacing them with one part equal to their sum. In particular, $b_1 + b_2 + b_3 \le 1$, except that $b_3 > 1$ is possible when $b_1 = b_2 = 0$.

For each family, we use induction on $s+t$. In each family, the claim trivially holds for the degenerate case $s = 0$. We first verify splittability for the instances with $s \geq 1$ and smallest t. Subsequently, we may assume that $s \geq 1$ and that t exceeds the smallest value, which allows us by the induction hypothesis to assume when $l = jt$ that there is no j-block. In the last family, the gap for $l = 11$ forces us to consider $(60s, t)$ separately for $t \in \{8, 9, 10\}$.

Case 1: $(20s, 3t)$ *with* $t \geq 3$*,* so $\sigma(M) \geq 29$. For $t = 3$ and $s \geq 1$, suppose that M contains no 20-block or 9-block. We have $b_3 \leq 2$ and $4b_4 + 5b_5 \leq 16$, since $b_4 b_5 = 0$. Hence $\sigma(M) \leq 22$, a contradiction. For larger t, we may assume $b_3 = 0$, $b_4 \leq 4$, and $b_5 \leq 3$. Also $b_1 + 2b_2 \leq 2$. Hence $\sigma(M) \leq 33$, which leaves only the case $(20, 12)$. This arises only when $(b_1, b_4, b_5) = (1, 4, 3)$, but then three 4-blocks yields the split.

Case 2: $(15s, 4t)$ *with* $t \geq 2$*,* so $\sigma(M) \geq 23$. For $t = 2$ and $s \geq 1$, suppose that M contains no 15-block or 8-block. We have $b_4 \leq 1$ and $3b_3 + 5b_5 \leq 12$, since $b_3 b_5 = 0$. Hence $\sigma(M) \leq 16$, a contradiction. For larger t, we may assume $b_4 = 0$, $b_3 \leq 4$, and $b_5 \leq 2$. Hence $\sigma(M) \leq 22$, a contradiction.

Case 3: $(12s, 5t)$ *with* $t \geq 2$*,* so $\sigma(M) \geq 22$. For $t = 2$ and $s \geq 1$, suppose that M contains no 12-block or 10-block. We have $b_5 \leq 1$ and $3b_3 + 4b_4 \leq 11$. Hence $\sigma(M) \leq 16$, a contradiction. For larger t, we may assume $b_5 = 0$, $b_4 \leq 2$, and $b_3 \leq 3$. Hence $\sigma(M) \leq 17$, a contradiction.

Case 4: $(60s, l)$ *with* $l \geq 12$*.* Since $D(5) = 5$, it suffices to obtain the split for $12 \leq l \leq 16$. For $l \in \{12, 15\}$, apply Case 1 above. For $(60s, 13)$, any single part in $\{1, 3, 4, 5\}$ reduces the search to an earlier case; since $60s + 13$ is odd, the parts cannot all equal 2. Similarly, for $(60s, 14)$ it suffices to have a part in $\{1, 2, 4, 5\}$, and the parts cannot all equal 3. □

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