

On the chaos bound in rotating black holes

Viktor Jahnke,^a Keun-Young Kim^a and Junggi Yoon^b

^a*School of Physics and Chemistry, Gwangju Institute of Science and Technology,
123 Cheomdan-gwagiro, Gwangju 61005, Korea*

^b*School of Physics, Korea Institute for Advanced Study,
85 Hoegiro Dongdaemun-gu, Seoul 02455, Republic of Korea*

E-mail: viktorjahnke@gist.ac.kr, fortoe@gist.ac.kr,
junggiyoon@kias.re.kr

ABSTRACT: We study out-of-time-order correlators (OTOCs) of rotating BTZ black holes using two different approaches: the elastic eikonal gravity approximation, and the Chern-Simons formulations of 3-dimensional gravity. Within both methods the OTOC is given as a sum of two contributions, corresponding to left and right moving modes. The contributions have different Lyapunov exponents, $\lambda_L^\pm = \frac{2\pi}{\beta} \frac{1}{1 \mp \ell\Omega}$, where Ω is the angular velocity and ℓ is the AdS radius. Since $\lambda_L^- \leq \frac{2\pi}{\beta} \leq \lambda_L^+$, there is an apparent contradiction with the chaos bound. We discuss how the result can be made consistent with the chaos bound if one views the parameters $\beta_\pm = \beta(1 \mp \ell\Omega)$ as the effective inverse temperatures of the left and right moving modes.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence

ARXIV EPRINT: [1903.09086](https://arxiv.org/abs/1903.09086)

Contents

1	Introduction	1
2	The rotating BTZ black hole	3
2.1	Embedding coordinates and bulk-to-boundary propagators	4
2.2	The dual CFT description	6
3	Eikonal approximation	6
3.1	Wave functions	9
3.2	The eikonal phase	10
3.3	Evaluating the integrals	11
4	Chaos in the Chern-Simons formulation of AdS₃ gravity	13
4.1	Review of Chern-Simons formulation of AdS ₃ gravity	13
4.2	On-shell action of asymptotic AdS solutions	16
4.3	Four point function in Euclidean correlators	18
4.4	Boyer-Lindquist co-rotating frame	24
4.5	OTOCs from analytic continuation	25
5	Discussion	26
A	Finite residual gauge transformations	29

1 Introduction

In recent years, the gauge-gravity duality [1–3] has provided remarkable insights into the nature of quantum chaos.¹ An important lesson from the holographic approach to quantum chaos [6–9] is the fact that the onset of chaos can be efficiently diagnosed with the so-called out-of-time-order correlators (OTOCs)

$$F(t, \vec{x}) = \langle V(0)W(t, \vec{x})V(0)W(t, \vec{x}) \rangle, \tag{1.1}$$

which, for chaotic systems, are expected to vanish at later times for almost any local operators V and W . In holographic systems at finite temperature, the OTOC has a very simple form

$$\frac{\langle V(0)W(t, \vec{x})V(0)W(t, \vec{x}) \rangle}{\langle V(0)V(0) \rangle \langle W(t, \vec{x})W(t, \vec{x}) \rangle} = 1 - \varepsilon_{\Delta_V \Delta_W} \exp \left[\lambda_L \left(t - t_* - \frac{|\vec{x}|}{v_B} \right) \right], \tag{1.2}$$

where the multiplicative factor, $\varepsilon_{\Delta_V \Delta_W}$, contains information about the operators V and W , while the exponential corresponds to an universal piece, and it is characterized by only

¹These recent developments were reviewed in [4, 5].

three parameters: the Lyapunov exponent λ_L , the scrambling time t_* , and the butterfly velocity v_B . All these quantities are determined from the geometry close to the black hole horizon. For black holes, one generically expects $\lambda_L = \frac{2\pi}{\beta}$, where β is the system's inverse temperature. Surprisingly, the above value provides an upper bound (the *chaos bound*) for the Lyapunov exponent in generic large- N systems, namely [10]

$$\lambda_L \leq \frac{2\pi}{\beta}. \quad (1.3)$$

This highlights a special property of holographic systems which are dual to black holes - they saturate the chaos bound. This finding generated a lot of excitement in the community, leading to the speculation that the saturation of the chaos bound could be a sufficient condition for the existence of Einstein gravity dual [10]. However, despite being a necessary condition, the saturation of the chaos does not guarantee the existence of dual description purely in terms of Einstein gravity [11–13]. The observation that the SYK model saturates the chaos bound put this system into evidence as a prototypical example of a simple model for holography. Several aspects of this system can be captured by a 2-dimensional Einstein-dilation theory, and this has generated a lot of research within the framework of NAdS₂/NCFT₁ and related areas. See, for instance, [14–26].

One expects the saturation of the chaos bound to be a property of quite generic black holes [10]. However, some recent studies suggest that rotating black holes have two Lyapunov exponents, one of which does not saturate the chaos bound, while the other violates it [27, 28]. Both these works consider the 3-dimensional rotating BTZ black hole. In [27], the author finds an effective action for the boundary degrees of freedom and studies OTOCs for this effective theory. He finds that the OTOC is controlled by two modes (left and right moving modes), with Lyapunov exponents $\lambda_L^\pm = \frac{2\pi}{\beta} \frac{1}{1 \mp \ell\Omega}$, where $\ell\Omega$ is the chemical potential for angular momentum. In [28], the author studies the disruption of the two-sided mutual information both in the CFT and in the bulk. In his particular configuration, the onset of chaos is controlled by λ_L^- , which indicates a non-saturation of the chaos bound. Moreover, the aforementioned works seem to be in contradiction with [29], in which the author studies chaos for the rotating BTZ black holes using the geodesic approximation and finds a saturation of the chaos bound.

In this paper, we revisit the calculation of OTOCs for rotating BTZ black holes using two different methods. We first use the elastic eikonal gravity approximation [9], in which the OTOCs have a very vivid holographic representation in terms of a high energy shock wave collision near the black hole's bifurcation surface. We then study chaos using the Chern-Simons formulation of 3-dimensional gravity. We show that the dynamics of the boundary degrees of freedom is governed by two copies of a Schwarzian-like action and derive the OTOCs from the analytic continuation of the euclidean 4-point. We compare both approaches with each other and with previous results in the literature, and discuss an apparent violation of the chaos bound in this system.

The paper is organized as follows. In section 2 we review the rotating BTZ black hole geometry and discuss a few aspects of the corresponding CFT dual description. In section 3 we compute OTOCs using the gravity eikonal approximation. In section 4 we

review the Chern-Simons formulation of 3-dimensional gravity and we derive an effective action for the boundary degrees of freedom. We then evaluate OTOCs from the on-shell action via analytic continuation. We discuss our results in section 5 and relegate some technical details to the appendix A.

2 The rotating BTZ black hole

In this section we briefly review the rotating BTZ geometry [30, 31]. This is a solution of Einstein gravity in 2+1 dimensions with a negative cosmological constant $\Lambda = -1/\ell^2$. In terms of *Schwarzschild coordinates* (t, r, φ) , the metric reads

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\varphi - \frac{r_+ r_-}{\ell r^2} dt \right)^2, \quad (2.1)$$

where φ is periodic with period 2π and the blackening factor has the form

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2}. \quad (2.2)$$

Here r_+ and r_- are the radii of the outer and inner horizons, respectively. The black hole's mass M , angular momentum J , and temperature T are determined from r_{\pm}

$$M = \frac{r_+^2 + r_-^2}{\ell^2}, \quad J = \frac{2r_+ r_-}{\ell}, \quad \beta = \frac{1}{T} = \frac{2\pi \ell^2 r_+}{r_+^2 - r_-^2}. \quad (2.3)$$

Since we will be interested in the near-horizon geometry, it is convenient to work with *co-rotating coordinates* (t, r, ϕ) , with the new angular variable defined as

$$\phi \equiv \varphi - \Omega t, \quad (2.4)$$

where

$$\Omega = \frac{r_-}{\ell r_+}, \quad (2.5)$$

is the angular velocity of the outer horizon. In these coordinates the metric takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(N^\phi(r)dt + d\phi \right)^2, \quad (2.6)$$

where

$$N^\phi(r) = \frac{r_-}{r_+} \frac{r^2 - r_+^2}{\ell r^2}. \quad (2.7)$$

In the near-horizon limit, $N^\phi(r)$ vanishes and the metric becomes diagonal.

Figure 1 shows the Penrose diagram for the maximally extended BTZ geometry. Here we follow [32] and denote the left and right exterior regions as 1_{+-} and 1_{++} , while the future and past interiors are denoted as 2_{++} and 2_{+-} , respectively. In figure 1 we only shows the region of interest. The same structure repeats itself in the vertical direction.

For later purposes, we introduce Kruskal coordinates for the region 1_{++}

$$U = -e^{-\kappa(t-r_*)}, \quad V = e^{\kappa(t+r_*)}, \quad \kappa = \frac{r_+^2 - r_-^2}{\ell^2 r_+}, \quad (2.8)$$

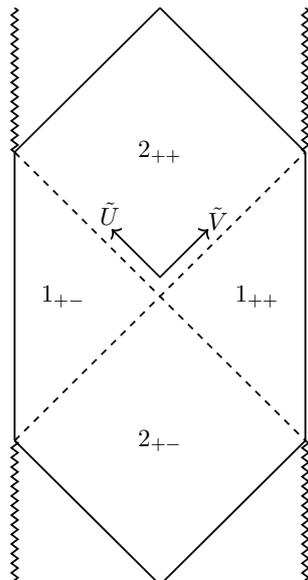


Figure 1. Penrose diagram for the rotating BTZ black hole. The figure only shows the region of interest. The same pattern repeats itself indefinitely above and below [31]. The regions 1_{++} and 1_{+-} denote the right and left exterior regions, while the regions 2_{+-} and 2_{++} denote the past and future interiors. The coordinates (\tilde{U}, \tilde{V}) are defined as in [33]: $V = e^{\frac{\pi}{2}} \tan\left(\frac{\tilde{V}}{2}\right)$ and $U = e^{\frac{\pi}{2}} \tan\left(\frac{\tilde{U}}{2}\right)$, with $\tilde{U}, \tilde{V} \in [-\pi, \pi]$.

where the tortoise coordinate r_* is [29]

$$r_* = \frac{1}{2\kappa} \log \left(\frac{\sqrt{r^2 - r_-^2} - \sqrt{r_+^2 - r_-^2}}{\sqrt{r^2 - r_-^2} + \sqrt{r_+^2 - r_-^2}} \right). \quad (2.9)$$

In these coordinates the metric takes the form

$$ds^2 = \frac{-4\ell^2 dU dV - 4\ell r_-(U dV - V dU) d\phi + [(1 - UV)^2 r_+^2 + 4UV r_-^2] d\phi^2}{(1 + UV)^2} \quad (2.10)$$

2.1 Embedding coordinates and bulk-to-boundary propagators

The AdS_3 space is defined as the hyperboloid $-T_1^2 - T_2^2 + X_1^2 + X_2^2 = -\ell^2$ embedded in a space with metric $ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + dX_2^2$. The rotating BTZ geometry can be thought of as piece of pure AdS_3 , with identifications in the angular coordinates, namely $\phi \sim \phi + 2\pi$. This fact allows us to compute bulk-to-boundary propagators for this geometry.

We start by defining embedding coordinates for the region 1_{++} . In terms of co-rotating coordinates, the embedding coordinates take the form

$$T_1 = \ell \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \sinh \left(\kappa t - \frac{r_-}{\ell} \phi \right), \quad (2.11)$$

$$T_2 = \ell \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \cosh \left(\frac{r_+}{\ell} \phi \right), \quad (2.12)$$

$$X_1 = \ell \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \cosh\left(\kappa t - \frac{r_-}{\ell} \phi\right), \quad (2.13)$$

$$X_2 = \ell \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \sinh\left(\frac{r_+}{\ell} \phi\right). \quad (2.14)$$

In terms of Kruskal coordinates, we have

$$T_1 = \ell \frac{V+U}{1+UV} \cosh\left(\frac{r_-}{\ell} \phi\right) - \ell \frac{V-U}{1+UV} \sinh\left(\frac{r_-}{\ell} \phi\right), \quad (2.15)$$

$$T_2 = \ell \frac{1-UV}{1+UV} \cosh\left(\frac{r_+}{\ell} \phi\right), \quad (2.16)$$

$$X_1 = \ell \frac{V-U}{1+UV} \cosh\left(\frac{r_-}{\ell} \phi\right) - \ell \frac{V+U}{1+UV} \sinh\left(\frac{r_-}{\ell} \phi\right), \quad (2.17)$$

$$X_2 = \ell \frac{1-UV}{1+UV} \sinh\left(\frac{r_+}{\ell} \phi\right). \quad (2.18)$$

By replacing the above formulas in the metric $ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + dX_2^2$, we can recover the formulas (2.6) and (2.10).

Having defined the embedding coordinates, we can now compute the geodesic distance d between two points $p = (T_1, T_2, X_1, X_2)$ and $p' = (T'_1, T'_2, X'_1, X'_2)$ as

$$\cosh\left(\frac{d}{\ell}\right) = \frac{1}{\ell^2} (T_1 T'_1 + T_2 T'_2 - X_1 X'_1 - X_2 X'_2). \quad (2.19)$$

We will use the above formulas to compute bulk-to-boundary propagators. For our calculation, it will be convenient to write the boundary point in terms of co-rotating coordinates $p' = (t, r, \phi')$ and the bulk point in terms of Kruskal coordinates $p = (U, V, \phi)$. The geodesic distance between such points can be written as

$$\begin{aligned} \cosh\left(\frac{d}{\ell}\right) &= \frac{r_\infty (r_+^2 - r_-^2)^{-1/2}}{1+UV} \\ &\times \left[U e^{\kappa t' + \frac{r_-}{\ell} \Delta\phi} - V e^{-(\kappa t' + \frac{r_-}{\ell} \Delta\phi)} + (1-UV) \cosh\left(\frac{r_+}{\ell} \Delta\phi\right) \right], \end{aligned} \quad (2.20)$$

where $\Delta\phi = \phi - \phi'$ and r_∞ is an UV cutoff defining the radial position of the boundary point.

From the above expression we can obtain the bulk-to-boundary propagator as proportional to $r_\infty^\Delta (\cosh(\frac{d}{\ell}))^{-\Delta}$, i.e.,

$$\begin{aligned} \langle \Phi(U, V, \phi) \mathcal{O}(t', \phi') \rangle &= \frac{c}{1+UV} \\ &\times \left[U e^{\kappa t' + \frac{r_-}{\ell} \Delta\phi} - V e^{-(\kappa t' + \frac{r_-}{\ell} \Delta\phi)} + (1-UV) \cosh\left(\frac{r_+}{\ell} \Delta\phi\right) \right]^{-\Delta}, \end{aligned} \quad (2.21)$$

where c is a constant that depends on the operator \mathcal{O} . Since we are only going to compute the bulk-to-boundary propagators when the bulk point is either at $U = 0$ or at $V = 0$, we

can simplify the above expression even further and write

$$\langle \Phi(U, V, \phi) \mathcal{O}(t', \phi') \rangle = c \left[U e^{\kappa t' + \frac{r_-}{\ell} \Delta \phi} - V e^{-(\kappa t' + \frac{r_-}{\ell} \Delta \phi)} + \cosh \left(\frac{r_+}{\ell} \Delta \phi \right) \right]^{-\Delta}. \quad (2.22)$$

Finally, to describe the black hole, we need to take into account the periodicity of the angular variable. This is done by shifting the angular coordinate by integer multiples of 2π and adding an infinite sum in front of the propagator as² [34]

$$\langle \Phi(U, V, \phi) \mathcal{O}(t', \phi') \rangle = c \sum_{n=-\infty}^{n=\infty} \left[U e^{\kappa t' + \frac{r_-}{\ell} \Delta \phi_n} - V e^{-(\kappa t' + \frac{r_-}{\ell} \Delta \phi_n)} + \cosh \left(\frac{r_+}{\ell} \Delta \phi_n \right) \right]^{-\Delta}, \quad (2.23)$$

where $\Delta \phi_n = \Delta \phi + 2\pi n$.

2.2 The dual CFT description

The rotating BTZ black hole is dual to a CFT with a chemical potential for angular momentum. The maximally extended rotating black hole is dual to thermofield double (TFD) state made of two such CFTs

$$|\text{TFD}\rangle_{t=0} = \frac{1}{Z(\beta, \Omega)^{1/2}} \sum_n e^{-\beta(E_n + \Omega J_n)/2} |E_n, J_n\rangle_L \otimes |E_n, J_n\rangle_R, \quad (2.24)$$

$$Z(\beta, \Omega) = \text{Tr} e^{-\beta(H + \Omega J)}, \quad (2.25)$$

where each CFT has Hamiltonian H and angular momentum J , with eigenvalues E_n and J_n , respectively. Here $Z(\beta, \Omega)$ is the grand canonical partition function.

We can also decompose the system into left and right moving modes, and write the TFD state as [35]

$$|\text{TFD}\rangle_{t=0} = \frac{1}{Z(\beta_-, \beta_+)^{1/2}} \sum_n e^{-(\beta_+ E_n^{(-)} + \beta_- E_n^{(+)})/2} |E_n^{(-)}\rangle_L \otimes |E_n^{(+)}\rangle_R, \quad (2.26)$$

$$Z(\beta, \Omega) = \text{Tr} e^{-\beta_- H_+ - \beta_+ H_-}, \quad (2.27)$$

where the left/right operators $H_{\pm} = \frac{H \pm J}{2}$ have eigenvalues $E_n^{(\pm)}$, and the temperatures of the left/right moving modes are

$$\beta_{\pm} = \beta(1 \mp \ell \Omega). \quad (2.28)$$

3 Eikonal approximation

In this section we compute the out-of-time-order correlator (OTOC)

$$F = \langle \text{TFD} | V_{\phi_1}(t_1) W_{\phi_2}(t_2) V_{\phi_3}(t_3) W_{\phi_4}(t_4) | \text{TFD} \rangle, \quad (3.1)$$

using the holographic prescription given by Shenker and Stanford (S&S) in [9]. Here, to avoid clutter, the subscripts ϕ_1, ϕ_2, \dots are used to denote the four angles. We consider all

²This procedure is known as the method of the images.

the operators acting on the right side of the geometry.³ For simplicity, we also consider the operators V and W to be single-trace operators, because in this case the corresponding bulk-to-boundary propagators have a simple form. If the t'_i s are all real, the above OTOC diverges. A simple way to regularize this divergence is to consider imaginary times⁴

$$\begin{aligned} t_1 &= -t/2 + i\epsilon_1, & t_3 &= -t/2 + i\epsilon_3, \\ t_2 &= t/2 + i\epsilon_2, & t_4 &= t/2 + i\epsilon_4. \end{aligned} \tag{3.2}$$

The basic idea of the S&S approach is to view the OTOC as a scattering amplitude

$$F = \langle \text{out} | \text{in} \rangle, \tag{3.3}$$

where the ‘in’ and ‘out’ states are given by

$$\begin{aligned} |\text{in}\rangle &= V_{\phi_3}(t_3)W_{\phi_4}(t_4)|\text{TFD}\rangle, \\ |\text{out}\rangle &= W_{\phi_2}(t_2)^\dagger V_{\phi_1}(t_1)^\dagger|\text{TFD}\rangle. \end{aligned} \tag{3.4}$$

The above states have a simple description in the bulk in terms of two-particle states. The V -operator creates a bulk particle, which we call the V -particle, while the particle created by the W -operator is called the W -particle. For large enough t these particles will be highly boosted with respect to the $t = 0$ slice of the geometry. The W -particle will have a very large momentum in the V -direction, while the V -particle will have a very large momentum in the U -direction. As explained in [9], the state $V_{\phi_3}(t_3)W_{\phi_4}(t_4)|\text{TFD}\rangle$ can be thought of as an ‘in’ state, in which the particle’s wave functions is represented at some early time (before the collision), while the state $W_{\phi_2}(t_2)^\dagger V_{\phi_1}(t_1)^\dagger|\text{TFD}\rangle$ can be thought of as an ‘out’ state, with the particle’s wave functions represented at some late time (after the collision). See figure 2.

For convenience, we choose to represent the V -particle in the $U = 0$ slice of the geometry, and we expand the corresponding state in the basis $|p^U, \phi'\rangle$ of well defined momentum and angular position. Likewise, we represent the W -particle in the $V = 0$ slice of the geometry, in the basis $|p^V, \phi'\rangle$. The basis vectors have the following normalization⁵

$$\langle p^U, \phi' | q^U, \phi'' \rangle = \frac{A_0^2 p^U}{\pi r_+} \delta(p^U - q^U) \delta(\phi' - \phi''), \tag{3.5}$$

with a similar formula for $|q^V, \phi'\rangle$. Here A_0 is defined by $A(0)$ in (3.20).

The ‘in’ state can then be written as

$$V_{\phi_3}(t_3)W_{\phi_4}(t_4)|\text{TFD}\rangle = \int d\phi'_3 d\phi'_4 \int dp_3^U dp_3^V \psi_3(p_3^U, \phi'_3) \psi_4(p_4^V, \phi'_4) |p_3^U, \phi'_3\rangle \otimes |p_4^V, \phi'_4\rangle, \tag{3.6}$$

³Two-sided configurations, like, for instance, $\langle V_L W_R V_L W_R \rangle$, can be obtained from the one-sided OTOC by analytic continuation.

⁴Another possibility is smear the operators out in a time scale of order β .

⁵This normalization can be fixed by considering the overlap of two Klein-Gordon wave functions. For instance, in the $U = 0$ slice of the geometry, the overlap between ψ and Φ reads $\langle \psi | \Phi \rangle = (2ir_+) \int dV d\phi \psi^* \partial_V \Phi$. By writing this overlap in momentum space one can obtain the normalization given in (3.5).

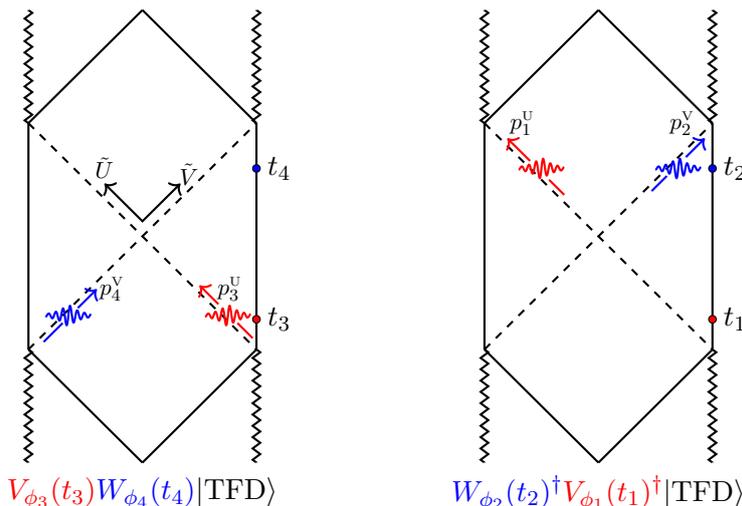


Figure 2. *Left:* the ‘in’ state $V_{\phi_3}(t_3)W_{\phi_4}(t_4)|\text{TFD}\rangle$ represented in a bulk spatial slice that touches the right boundary at time t_3 . *Right:* the ‘out’ state $W_{\phi_2}(t_2)^\dagger V_{\phi_1}(t_1)^\dagger|\text{TFD}\rangle$ represented in a bulk slice that touches the right boundary at time t_2 .

while the ‘out’ state reads

$$V_{\phi_1}(t_1)^\dagger W_{\phi_2}(t_2)^\dagger|\text{TFD}\rangle = \int d\phi_1' d\phi_2' \int dp_1^U dp_2^V \psi_1(p_1^U, \phi_1') \psi_2(p_2^V, \phi_2') |p_1^U, \phi_1'\rangle \otimes |p_2^V, \phi_2'\rangle. \quad (3.7)$$

The wave functions ψ_i appearing in the above formulas are Fourier transforms of bulk-to-boundary propagators

$$\psi_1(p^U, \phi') = \int dV e^{iA_0 p^U V} \langle \Phi_V(U, V, \phi') V_{\phi_1}(t_1)^\dagger \rangle|_{U=0}, \quad (3.8)$$

$$\psi_2(p^V, \phi') = \int dU e^{iA_0 p^V U} \langle \Phi_W(U, V, \phi') W_{\phi_2}(t_2)^\dagger \rangle|_{V=0}, \quad (3.9)$$

$$\psi_3(p^U, \phi') = \int dV e^{iA_0 p^U V} \langle \Phi_V(U, V, \phi') V_{\phi_3}(t_3) \rangle|_{U=0}, \quad (3.10)$$

$$\psi_4(p^V, \phi') = \int dU e^{iA_0 p^V U} \langle \Phi_W(U, V, \phi') W_{\phi_4}(t_4) \rangle|_{V=0}, \quad (3.11)$$

where Φ_V and Φ_W are bulk fields dual to the operators V and W , which are represented via the ‘extrapolate’ dictionary.

In the eikonal approximation, the ‘in’ and ‘out’ are related by a phase shift

$$(|p_1^U, \phi_1'\rangle \otimes |p_2^V, \phi_2'\rangle)_{\text{out}} \approx e^{i\delta(s, \phi_1' - \phi_2')} (|p_1^U, \phi_1'\rangle \otimes |p_2^V, \phi_2'\rangle)_{\text{in}} + |\text{inelastic}\rangle, \quad (3.12)$$

where $\delta(s, \phi_1' - \phi_2')$ is the so-called *Eikonal phase* and $s = (p_1 + p_2)^2 = 2A_0 p^U p^V$ is a Mandelstam variable. The state $|\text{inelastic}\rangle$ represents an inelastic contribution that is orthogonal to all two-particle ‘in’ states. The physics behind the eikonal approximation is the following: since the particles collide at very high energy, the interaction is dominated by the graviton exchange, and the sum of the corresponding ladder diagrams can be described by a simple phase shift [36, 37].

The OTOC (3.1) can then be written as

$$F = \frac{A_0^4}{\pi^2} \iint d\phi d\phi' \iint dp_1^U dp_2^V e^{i\delta(s, \phi - \phi')} \left[p_1^U \psi_1^*(p_1^U, \phi) \psi_3(p_1^U, \phi) \right] \left[p_2^V \psi_2^*(p_2^V, \phi') \psi_4(p_2^V, \phi') \right], \quad (3.13)$$

where $p_3^U = p_1^U$, $p_4^V = p_2^V$, $\phi_3 = \phi_1 \equiv \phi$ and $\phi_4 = \phi_2 \equiv \phi'$ due to (3.5). The information of ϕ_i and t_i with $i = 1, 2, 3, 4$ in (3.1) is denoted by the subscripts of ψ_i .

3.1 Wave functions

For the rotating BTZ black hole the bulk-to-boundary propagators in (3.8) and (3.9) are given by

$$\langle \Phi_V(0, V, \phi) V_{\phi_1}(t_1)^\dagger \rangle = c_V \sum_{n=-\infty}^{n=\infty} \left[-V e^{-(\kappa t_1^* + \frac{r_-}{\ell} \Delta\phi_n)} + \cosh\left(\frac{r_+}{\ell} \Delta\phi_n\right) \right]^{-\Delta_V}, \quad (3.14)$$

$$\langle \Phi_W(U, 0, \phi') W_{\phi_2}(t_2)^\dagger \rangle = c_W \sum_{n=-\infty}^{n=\infty} \left[U e^{\kappa t_2^* + \frac{r_-}{\ell} \Delta\phi'_n} + \cosh\left(\frac{r_+}{\ell} \Delta\phi'_n\right) \right]^{-\Delta_W}, \quad (3.15)$$

where

$$\Delta\phi_n = \phi - \phi_1 + 2\pi n, \quad \Delta\phi'_n = \phi' - \phi_2 + 2\pi n. \quad (3.16)$$

For the bulk-to-boundary propagators in (3.10) and (3.11) we only need to change $t \rightarrow t^*$ with

$$\Delta\phi_n = \phi - \phi_3 + 2\pi n, \quad \Delta\phi'_n = \phi' - \phi_4 + 2\pi n. \quad (3.17)$$

To avoid clutter, we use the same notation $\Delta\phi_n$ for ϕ_1 and ϕ_3 ($\Delta\phi'_n$ for ϕ_2 and ϕ_4).

The wave functions can then be written as

$$\begin{aligned} \psi_1(p^U, \phi) &= -\theta(p^U) \frac{2\pi i c_V}{\Gamma(\Delta_V)} \sum_{n=-\infty}^{n=+\infty} e^{\kappa t_1^* + \frac{r_-}{\ell} \Delta\phi_n} \\ &\quad \times \left(-i A_0 p^U e^{\kappa t_1^* + \frac{r_-}{\ell} \Delta\phi_n} \right)^{\Delta_V - 1} e^{i A_0 p^U \cosh\left(\frac{r_+}{\ell} \Delta\phi_n\right) e^{\kappa t_1^* + \frac{r_-}{\ell} \Delta\phi_n}}, \\ \psi_3(p^U, \phi) &= -\theta(p^U) \frac{2\pi i c_V}{\Gamma(\Delta_V)} \sum_{n=-\infty}^{n=+\infty} e^{\kappa t_3 + \frac{r_-}{\ell} \Delta\phi_n} \\ &\quad \times \left(-i A_0 p^U e^{\kappa t_3 + \frac{r_-}{\ell} \Delta\phi_n} \right)^{\Delta_V - 1} e^{i A_0 p^U \cosh\left(\frac{r_+}{\ell} \Delta\phi_n\right) e^{\kappa t_3 + \frac{r_-}{\ell} \Delta\phi_n}}, \\ \psi_2(p^V, \phi') &= \theta(p^V) \frac{2\pi i c_W}{\Gamma(\Delta_W)} \sum_{n=-\infty}^{n=+\infty} e^{-\kappa t_2^* - \frac{r_-}{\ell} \Delta\phi'_n} \\ &\quad \times \left(i A_0 p^V e^{\kappa t_2^* + \frac{r_-}{\ell} \Delta\phi'_n} \right)^{\Delta_W - 1} e^{-i A_0 p^V \cosh\left(\frac{r_+}{\ell} \Delta\phi'_n\right) e^{-\kappa t_2^* - \frac{r_-}{\ell} \Delta\phi'_n}}, \\ \psi_4(p^V, \phi') &= \theta(p^V) \frac{2\pi i c_W}{\Gamma(\Delta_W)} \sum_{n=-\infty}^{n=+\infty} e^{-\kappa t_4 - \frac{r_-}{\ell} \Delta\phi'_n} \\ &\quad \times \left(i A_0 p^V e^{\kappa t_4 + \frac{r_-}{\ell} \Delta\phi'_n} \right)^{\Delta_W - 1} e^{-i A_0 p^V \cosh\left(\frac{r_+}{\ell} \Delta\phi'_n\right) e^{-\kappa t_4 - \frac{r_-}{\ell} \Delta\phi'_n}}. \end{aligned} \quad (3.18)$$

3.2 The eikonal phase

The operators V and W introduce particles in the bulk, which we call the V -particle and the W -particle, respectively. In the elastic eikonal gravity approximation, the phase shift is basically given by the on-shell action of these particles, namely

$$\delta(s, \phi - \phi') = S_{\text{classical}}, \quad (3.19)$$

where $S_{\text{classical}}$ is simply the sum of the on-shell actions for the V -particles and W -particles. When the W is very late, and the V operator is very early, the corresponding bulk particles follow an almost null trajectory very close the black hole horizon. The W -particle moves along $U = 0$, while the V -particle moves along $V = 0$. The particles collide very close to the bifurcation surface, at $U = V = 0$, and the phase shift is only sensitive to this region of the geometry, where the metric assumes the approximate form

$$ds^2 = -2A(UV)dUdV + r^2(UV)d\phi^2. \quad (3.20)$$

In the following, we denote the horizon values by $A_0 \equiv A(0)$ and $r_+ \equiv r(0)$. In this particular configuration, the stress-energy of these particles takes the form

$$\begin{aligned} W\text{-particle} : T_{UU} &= \frac{A_0}{r_+} p_2^V \delta(U) \delta(\phi - \phi''), \\ V\text{-particle} : T_{VV} &= \frac{A_0}{r_+} p_1^U \delta(V) \delta(\phi - \phi'). \end{aligned} \quad (3.21)$$

where ϕ' and ϕ'' denote the position of the sources. The back-reaction of these particles in the geometry takes a simple shock wave form. For instance, the backreaction of the W -particles reads

$$ds^2 \rightarrow ds^2 + h_{UU}dU^2, \quad h_{UU} = 16\pi G_N r_+ A_0 p_2^V \delta(U) f(\phi - \phi''), \quad (3.22)$$

while the back-reaction of the V -particle reads

$$ds^2 \rightarrow ds^2 + h_{VV}dV^2, \quad h_{VV} = 16\pi G_N r_+ A_0 p_1^U \delta(V) f(\phi - \phi'). \quad (3.23)$$

In the above formulas, $f(\phi)$ is the shock wave transverse profile. For a rotating BTZ black hole, $f(\phi)$ is a solution of the equation⁶

$$\ell^2 f''(\phi) - 2\ell r_- f'(\phi) - (r_+^2 - r_-^2) f(\phi) = \delta(\phi). \quad (3.24)$$

The most general solution of the above equation has the form

$$f(\phi) = c_1 e^{(r_+ + r_-)\phi/\ell} + c_2 e^{-(r_+ - r_-)\phi/\ell}, \quad (3.25)$$

where c_1 and c_2 are constants and we assumed $\phi > 0$.⁷

⁶This equation is obtained from the linearized Einstein's equations. As a result, our shock wave solutions are only valid at the perturbative level. However, the phase shift is only sensitive to the region very close to the bifurcation surface, where the above shock wave solutions are exact solutions of Einstein's equations.

⁷For a more detailed discussion of this solution and the constants, see [38]. Notice, however, that our co-rotating coordinate differ from the one used in that reference by a minus sign, i.e., $\phi_{\text{here}} = -\phi_{\text{there}}$.

The on-shell classical action for the W -particles and V -particles can be written as [9]

$$S_{\text{classical}} = \frac{1}{2} \int d^3x \sqrt{-g} h_{UU} T^{UU}. \quad (3.26)$$

Note that, while h_{UU} refers to the W -particle, the stress-tensor $T^{UU} = g^{UV} g^{UV} T_{VV}$ refers to the V -particle. Substituting the expression for the stress-energy tensors and the corresponding back-reactions, one finds

$$\delta(s, \phi' - \phi'') = 4\pi G_N r_+ s f(\phi' - \phi''), \quad s = 2A_0 p_1^U p_2^V. \quad (3.27)$$

3.3 Evaluating the integrals

Inserting the above wave functions in the formula for the OTOC (3.13), we find

$$\begin{aligned} F &= \left(\frac{4\pi c_V c_W}{\Gamma(\Delta_V)\Gamma(\Delta_W)} \right)^2 A_0^{2(\Delta_V + \Delta_W)} \\ &\times \sum_{n_1, n_2, n_3, n_4} \int d\phi d\phi' dp^U dp^V e^{i\delta(s, \phi - \phi')} (p^U)^{2\Delta_V - 1} (p^V)^{2\Delta_W - 1} \frac{e^{\Delta_V \kappa(t_1 + t_3)}}{e^{\Delta_W \kappa(t_2 + t_4)}} \\ &\times \frac{e^{\Delta_V \frac{r_-}{\ell} (\Delta\phi_{n_1} + \Delta\phi_{n_3})}}{e^{\Delta_W \frac{r_-}{\ell} (\Delta\phi'_{n_2} + \Delta\phi'_{n_4})}} \frac{e^{iA_0 p^U \left[\cosh\left(\frac{r_+}{\ell} \Delta\phi_{n_3}\right) e^{\kappa t_3 + \frac{r_-}{\ell} \Delta\phi_{n_3}} - \cosh\left(\frac{r_+}{\ell} \Delta\phi_{n_1}\right) e^{\kappa t_1 + \frac{r_-}{\ell} \Delta\phi_{n_1}} \right]}}{e^{iA_0 p^V \left[\cosh\left(\frac{r_+}{\ell} \Delta\phi'_{n_4}\right) e^{\kappa t_4 + \frac{r_-}{\ell} \Delta\phi'_{n_4}} - \cosh\left(\frac{r_+}{\ell} \Delta\phi'_{n_2}\right) e^{\kappa t_2 + \frac{r_-}{\ell} \Delta\phi'_{n_2}} \right]}}, \end{aligned} \quad (3.28)$$

where $\Delta\phi_{n_i} = \phi - \phi_1 + 2\pi n_i$ and $\Delta\phi'_{n_i} = \phi' - \phi_2 + 2\pi n_i$.

The OTOC can be divided into two contributions

$$F = F_0 + F_{n_1, n_2, n_3, n_4}, \quad (3.29)$$

where F_0 corresponds to the term where $n_1 = n_2 = n_3 = n_4 = 0$, and F_{n_1, n_2, n_3, n_4} corresponds to the remainder terms. The dominant contribution comes from F_0

$$\begin{aligned} F_0 &= \left(\frac{4\pi c_V c_W}{\Gamma(\Delta_V)\Gamma(\Delta_W)} \right)^2 A_0^{2(\Delta_V + \Delta_W)} \int d\phi d\phi' dp^U dp^V e^{i\delta(s, \phi - \phi')} (p^U)^{2\Delta_V - 1} (p^V)^{2\Delta_W - 1} \\ &\times \frac{e^{\Delta_V \kappa(t_1 + t_3)}}{e^{\Delta_W \kappa(t_2 + t_4)}} \frac{e^{2\Delta_V \frac{r_-}{\ell} \Delta\phi}}{e^{2\Delta_W \frac{r_-}{\ell} \Delta\phi'}} \frac{e^{iA_0 p^U \cosh\left(\frac{r_+}{\ell} \Delta\phi\right) e^{\frac{r_-}{\ell} \Delta\phi} (e^{\kappa t_3} - e^{\kappa t_1})}}{e^{iA_0 p^V \cosh\left(\frac{r_+}{\ell} \Delta\phi'\right) e^{\frac{r_-}{\ell} \Delta\phi'} (e^{\kappa t_4} - e^{\kappa t_2})}}. \end{aligned} \quad (3.30)$$

The only ingredient missing to calculate F_0 is the phase shift $\delta(s, \phi - \phi')$, which we know to have the following form (see (3.27))

$$\delta(s, \phi - \phi') = 8\pi r_+ G_N p^U p^V f(\phi - \phi'), \quad (3.31)$$

where $f(\phi - \phi')$ is given by (3.25).

To evaluate the above integral it is convenient to introduce the new variables

$$\begin{aligned} p &= -iA_0 p^U (e^{\kappa t_3} - e^{\kappa t_1}), \\ q &= iA_0 p^V (e^{\kappa t_4} - e^{\kappa t_2}). \end{aligned} \quad (3.32)$$

By specifying the times as in (3.2), the integral becomes

$$F_0 = C \int d\phi d\phi' dp dq p^{2\Delta_V-1} q^{2\Delta_W-1} e^{i8\pi r_+ G_N A_0^{-1} \frac{pq}{\epsilon_{13}\epsilon_{24}} e^{\kappa t} f(\phi-\phi')} \times e^{\left[\frac{2r_-}{\ell} (\Delta_V \Delta\phi - \Delta_W \Delta\phi') - p \cosh\left(\frac{r_+}{\ell} \Delta\phi\right) e^{\frac{r_-}{\ell} \Delta\phi} - q \cosh\left(\frac{r_+}{\ell} \Delta\phi'\right) e^{\frac{r_-}{\ell} \Delta\phi'} \right]}, \quad (3.33)$$

where $\epsilon_{ij} = i(e^{\kappa\epsilon_i} - e^{\kappa\epsilon_j})$ and C is a constant given by

$$C = \frac{2\pi^2 c_V^2 c_W^2}{\Gamma(\Delta_V)^2 \Gamma(\Delta_W)^2} \left[\frac{1}{2 \sin\left(\frac{\epsilon_3 - \epsilon_1}{2}\right)} \right]^{2\Delta_V} \left[\frac{1}{2 \sin\left(\frac{\epsilon_4 - \epsilon_2}{2}\right)} \right]^{2\Delta_W}. \quad (3.34)$$

Without the phase shift, the above integral just gives $\langle VV \rangle \langle WW \rangle$. The integral can be evaluated in the limit $\Delta_W \gg \Delta_V \gg 1$ and the result reads⁸

$$\begin{aligned} \text{OTOC}(t, \phi_{12}) &= \frac{\langle V_{\phi_1}(t_1) W_{\phi_2}(t_2) V_{\phi_1}(t_3) W_{\phi_2}(t_4) \rangle}{\langle V_{\phi_1}(i\epsilon_1) V_{\phi_1}(i\epsilon_3) \rangle \langle W_{\phi_1}(i\epsilon_2) W_{\phi_4}(i\epsilon_4) \rangle} = \frac{1}{\left[1 + \frac{i16\pi G_N \Delta_W}{\ell^2 \epsilon_{13} \epsilon_{24}} e^{\kappa t} f(\phi_{12}) \right]^{\Delta_V}} \\ &\approx 1 + \frac{i16\pi G_N \Delta_V \Delta_W}{\ell^2 \epsilon_{13} \epsilon_{24}} e^{\kappa t} f(\phi_{12}), \end{aligned} \quad (3.35)$$

where $\phi_{12} \equiv \phi_1 - \phi_2$. Replacing $f(\phi)$ by (3.25) and using that

$$\frac{r_+ + r_-}{\ell} = \frac{2\pi}{\beta} \frac{\ell}{1 - \ell\Omega}, \quad \frac{r_+ - r_-}{\ell} = \frac{2\pi}{\beta} \frac{\ell}{1 + \ell\Omega}, \quad (3.36)$$

we can write the OTOC as

$$\text{OTOC}(t, \phi_{12}) \approx 1 + C_1 e^{\frac{2\pi}{\beta} \left(t + \frac{\ell\phi_{12}}{1 - \ell\Omega} \right)} + C_2 e^{\frac{2\pi}{\beta} \left(t - \frac{\ell\phi_{12}}{1 + \ell\Omega} \right)}, \quad (3.37)$$

where the constants C_1 and C_2 are proportional to $\Delta_V \Delta_W G_N$. These two solutions correspond to left and right moving modes. Both modes have a maximal Lyapunov exponent, $\lambda_L = \frac{2\pi}{\beta}$, but they have different butterfly velocities

$$\frac{v_B^\pm}{c} = 1 \pm \ell\Omega. \quad (3.38)$$

One of these butterfly velocities is larger than the speed of light when $\Omega > 0$.

In terms of the Schwarzschild angular coordinate, $\varphi_{12} = \phi_{12} + \Omega t_{12}$ (here $t_{12} = \text{Re}(t_1 - t_2) = -t$), we have

$$\text{OTOC}(t, \varphi_{12}) \approx 1 + C_1 e^{\frac{2\pi}{\beta_+} (t + \ell\varphi_{12})} + C_2 e^{\frac{2\pi}{\beta_-} (t - \ell\varphi_{12})}, \quad (3.39)$$

where

$$\beta_\pm = \beta(1 \mp \ell\Omega). \quad (3.40)$$

⁸Here we evaluate the integrals using the same tricks that were used in [9]. By writing the integrals in q and ϕ' as $\int dq d\phi' e^{-F(q, \phi')}$, we can check that the result is dominated by the region where $q \approx 2\Delta_W$ and $\phi' \approx \phi_2$. The integral in p can be done analytically, and the integral in ϕ can be done by a saddle point approximation.

In this case the two butterfly velocities are equal to the speed of light, but the two modes have different Lyapunov exponents

$$\lambda_L^\pm = \frac{2\pi}{\beta_\pm}, \tag{3.41}$$

which satisfy the property $\lambda_L^- \leq \lambda_L \leq \lambda_L^+$. Naively, this seems to indicate that one of the Lyapunov exponents is less than maximum, while the other one violates the chaos bound [10]. We come back to this issue in the discussion section.

4 Chaos in the Chern-Simons formulation of AdS₃ gravity

In the previous section, we calculated the Lyapunov exponent and the butterfly velocity using the gravity eikonal approximation. In this section, we use the $SL(2, \mathbb{C})$ Chern-Simons formulation of Euclidean AdS_3 gravity to derive the on-shell action of the boundary graviton. This is analogous to the Schwarzian on-shell action of the 2D Jackiw-Teitelboim model [18, 39–42] as well as the dimensional reduction from higher dimensional black holes [43–47]. Moreover, by using a metric-like formulation, [27] obtained a similar on-shell action, which is equivalent to our result up to quadratic level. Since we will evaluate the leading Lyapunov exponent in large $k_{cs} \sim c$ from the quadratic on-shell action, the leading Lyapunov exponent agrees with [27]. Also, the on-shell action from the Chern-Simons gravity and Lyapunov exponent thereof can easily be generalized for the case of higher spin gravity [48].

4.1 Review of Chern-Simons formulation of AdS₃ gravity

In this section, we review the $SL(2, \mathbb{C})$ Chern-Simons formulation of Euclidean AdS₃ (EAdS₃) gravity. In particular, we follow [49] in the choice of the boundary term and the corresponding boundary condition. The case of Lorentzian AdS_3 is similar, except that one has to choose carefully the “Kruskal-like” gauges for the eternal black hole [50].

The Euclidean solution can be obtained by analytic continuation of (2.1) to imaginary values of t and r_-

$$t = -it_E, \quad r_- = ir_E. \tag{4.1}$$

The Euclidean coordinates (t_E, φ) satisfy the periodic condition [34, 51]

$$(t_E, \varphi) \sim (t_E + \beta, \varphi + w), \tag{4.2}$$

where

$$w \equiv -i\beta\Omega = \beta \frac{r_E}{r_+ l}. \tag{4.3}$$

Note that w and r_E are taken to be real in the Euclidean black hole, but they become purely imaginary in the Lorentzian BTZ black hole. We define holomorphic and anti-holomorphic coordinates (z, \bar{z}) as

$$z \equiv \varphi + i \frac{t_E}{\ell}, \quad \bar{z} \equiv \varphi - i \frac{t_E}{\ell}. \tag{4.4}$$

The modular parameters τ and $\bar{\tau}$ are defined as

$$\tau \equiv w + \frac{i\beta}{l}, \quad \bar{\tau} \equiv w - \frac{i\beta}{l}. \tag{4.5}$$

In terms of the (z, \bar{z}) and $(\tau, \bar{\tau})$, the above periodic condition becomes

$$(z, \bar{z}) \sim (z + \tau, \bar{z} + \bar{\tau}). \quad (4.6)$$

In three dimensions, EAdS₃ gravity can be described by the Chern-Simons action with complex Lie algebra $SL(2, \mathbb{C})$ [52–55]

$$I_{\text{CS}} = \frac{ik_{\text{CS}}}{4\pi} \int_M \text{tr} [CS(A) - CS(\bar{A})], \quad (4.7)$$

where $k_{\text{CS}} = \frac{l}{4G}$ is Chern-Simons level and $CS(A)$ is defined by

$$CS(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A, \quad (4.8)$$

Here, A is the Chern-Simons connection, and \bar{A} is its conjugate, defined by

$$\bar{A} = -A^\dagger. \quad (4.9)$$

We use the coordinates (r, z, \bar{z}) where z and \bar{z} are defined by

$$z \equiv \varphi + i\frac{t_E}{l}, \quad \bar{z} \equiv \varphi - i\frac{t_E}{l}. \quad (4.10)$$

For EAdS₃ Chern-Simons gravity, the manifold in our consideration is a solid torus, and the modular parameter τ of the boundary torus gives the periodicity of z coordinate

$$z \sim z + 2\pi \sim z + \tau. \quad (4.11)$$

It is useful to fix the gauge as [49, 54–57]

$$A = b^{-1}(d + a_z dz + a_{\bar{z}} d\bar{z})b, \quad (4.12)$$

$$\bar{A} = b(d + \bar{a}_z dz + \bar{a}_{\bar{z}} d\bar{z})b^{-1}, \quad (4.13)$$

where $b(r)$ is defined by

$$b = e^{rL_0}. \quad (4.14)$$

Here, we define the $sl(2)$ generators $L_0, L_{\pm 1}$ as

$$L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \quad (4.15)$$

The asymptotic AdS₃ solution with a flat boundary metric in the Chern-Simons gravity was found to be [49, 56, 57]

$$A - A_{\text{AdS}}|_{\partial\mathcal{M}} \sim \mathcal{O}(1), \quad (4.16)$$

where A_{AdS} is the AdS₃ exact solution

$$A_{\text{AdS}} = b^{-1} \left(L_1 + \frac{1}{4} L_{-1} \right) b dz + b^{-1} db, \quad \bar{A}_{\text{AdS}} = b \left(L_{-1} + \frac{1}{4} L_1 \right) b^{-1} d\bar{z} + b db^{-1}. \quad (4.17)$$

Hence, the asymptotic AdS₃ condition implies that

$$a = L_1 dz + \dots, \quad \bar{a} = L_{-1} d\bar{z} + \dots. \quad (4.18)$$

The variation of the action (4.7) without the additional boundary term is given by

$$\delta I_{\text{CS}} = -\frac{ik_{\text{CS}}}{4\pi} \int_{\partial\mathcal{M}} \text{tr} [A \wedge \delta A - \bar{A} \wedge \delta \bar{A}]. \quad (4.19)$$

Hence, choosing the boundary condition $A_{\bar{z}} = \bar{A}_z = 0$ together with the gauge symmetry [56, 57], we can fix the gauge as follows

$$a = \left(L_1 - \frac{2\pi}{k_{\text{CS}}} \mathcal{L}(z) L_{-1} \right) dz, \quad (4.20)$$

$$\bar{a} = \left(L_{-1} - \frac{2\pi}{k_{\text{CS}}} \bar{\mathcal{L}}(\bar{z}) L_1 \right) d\bar{z}. \quad (4.21)$$

Now, we have to take the variation of the modular parameter τ because it can also be varied under the variation of the action. For this, we will summarize the $(\tau, \bar{\tau})$ formalism discussed in [49]. It is useful to introduce the new coordinates (w, \bar{w}) , defined by

$$z = \frac{1 - i\frac{\tau}{2\pi}}{2} w + \frac{1 + i\frac{\tau}{2\pi}}{2} \bar{w}, \quad (4.22)$$

which has a fixed periodicity

$$w \sim w + 2\pi \sim w + 2\pi i. \quad (4.23)$$

At a cost of fixing periodicity, the modular parameter appears in the boundary volume element as well as in the boundary metric

$$ds^2 = dzd\bar{z} = \left| \left(\frac{1 - i\frac{\tau}{2\pi}}{2} dw + \left(\frac{1 + i\frac{\tau}{2\pi}}{2} \right) d\bar{w} \right) \right|^2, \quad (4.24)$$

$$idw \wedge d\bar{w} = \frac{4\pi dz^2}{\text{Im}(\tau)}, \quad (4.25)$$

where $dz^2 = \frac{i}{2} dz \wedge d\bar{z}$. The key idea of [49] is to keep the (boundary) volume element of (w, \bar{w}) fixed under the variation. In (w, \bar{w}) coordinates, the differential form is not varied under the variation of the bulk action in (4.7). Instead, in returning to (z, \bar{z}) coordinates, the variation of $a_w, a_{\bar{w}}$ gives the variation of the modular parameter $\tau, \bar{\tau}$ because it appears in the transformation of $(a_w, a_{\bar{w}})$ into $(a_z, a_{\bar{z}})$

$$a_w = \left(\frac{1 - i\frac{\tau}{2\pi}}{2} \right) a_z + \left(\frac{1 - i\frac{\bar{\tau}}{2\pi}}{2} \right) a_{\bar{z}}. \quad (4.26)$$

For the variational principle, the authors of [49] choose the boundary term given by

$$I_b = -\frac{k_{\text{CS}}}{2\pi} \int_{\partial\mathcal{M}} d^2z \text{tr} ((a_z - 2L_1) a_{\bar{z}}) - \frac{k_{\text{CS}}}{2\pi} \int_{\partial\mathcal{M}} d^2z \text{tr} ((\bar{a}_{\bar{z}} - 2L_{-1}) \bar{a}_z), \quad (4.27)$$

and the variation of the total action $I_{\text{tot}} \equiv I_{\text{CS}} + I_b$ is found to be

$$\delta I_{\text{tot}} = -ik_{\text{CS}} \int_{\partial\mathcal{M}} \frac{d^2z}{2\pi\text{Im}(\tau)} \text{tr} \left[(a_z - L_1) \delta((\bar{\tau} - \tau)a_{\bar{z}}) + \left(\frac{a_z^2}{2} + a_z a_{\bar{z}} - \frac{\bar{a}_{\bar{z}}^2}{2} \right) \delta\tau \right. \\ \left. - (-\bar{a}_{\bar{z}} - L_{-1}) \delta((\bar{\tau} - \tau)\bar{a}_z) + \left(\frac{\bar{a}_{\bar{z}}^2}{2} + \bar{a}_{\bar{z}} \bar{a}_z - \frac{a_z^2}{2} \right) \delta\bar{\tau} \right]. \quad (4.28)$$

Hence, we can impose the boundary conditions

$$a_{\bar{z}} = \bar{a}_z = 0, \quad \delta\tau = \delta\bar{\tau} = 0. \quad (4.29)$$

With these boundary conditions, the on-shell action of I_{tot} is found to be

$$I_{\text{on-shell}} = \frac{ik_{\text{CS}}}{2\pi} \int \frac{d^2z}{\text{Im}(\tau)} \text{tr} \left[\frac{\tau}{2} a_z^2 - \frac{\bar{\tau}}{2} \bar{a}_{\bar{z}}^2 \right]. \quad (4.30)$$

Note that though the authors of [49] mainly analyzed a constant solution a and \bar{a} , they pointed out that their method could be applied to non-constant solutions. Furthermore, the $sl(2, \mathbb{C})$ Chern-Simons gravity is simple because an additional chemical potential is not necessary, as opposed to the case of higher spin gravity. In the next section, we will analyze the non-constant solutions and their on-shell action.

4.2 On-shell action of asymptotic AdS solutions

As in the previous section, the boundary condition in (4.29) allows us to fix the connections to be

$$a_z(z) = \begin{pmatrix} 0 & \frac{2\pi}{k_{\text{CS}}} \mathcal{L}(z) \\ 1 & 0 \end{pmatrix}, \quad \bar{a}_{\bar{z}}(\bar{z}) = \begin{pmatrix} 0 & -1 \\ -\frac{2\pi}{k_{\text{CS}}} \bar{\mathcal{L}}(\bar{z}) & 0 \end{pmatrix}. \quad (4.31)$$

Then, the on-shell action in (4.30) can be written as

$$I_{\text{on-shell}} = i \int \frac{d^2z}{\text{Im}(\tau)} [\tau \mathcal{L}(z) - \bar{\tau} \bar{\mathcal{L}}(\bar{z})]. \quad (4.32)$$

We consider non-constant connections, which are smoothly connected to a fixed constant solution, such as the BTZ black hole, i.e.,

$$a = h^{-1} a_{\text{BTZ}} h + h^{-1} dh, \quad (4.33)$$

where $h = h(z)$ is a holomorphic residual gauge transformation parameter that can be smoothly connected to identity, and the constant solution a_{BTZ} is given by

$$a_{\text{BTZ}} = \begin{pmatrix} 0 & \frac{2\pi}{k_{\text{CS}}} \mathcal{L}_0 \\ 1 & 0 \end{pmatrix}, \quad (4.34)$$

with \mathcal{L}_0 being a constant. Such a smooth residual gauge transformation does not change the holonomy (up to a similarity transformation)

$$\text{Hol}_{\mathbb{C}}(A) \equiv \mathcal{P} \exp \left[- \int_{\mathcal{C}} A \right] = b^{-1} h^{-1} e^{-\omega} h b. \quad (4.35)$$

The smoothness of a holonomy along the contractible cycle of solid torus requires the holonomy to belong to the center of the gauge group [58–60]. For the BTZ black hole, the Euclidean time circle corresponds to the contractible cycle, and the smoothness condition implies that

$$\mathfrak{w} = \tau a_z + \bar{\tau} a_{\bar{z}} = u^{-1}(2\pi i L_0)u, \tag{4.36}$$

for some matrix u . Hence, by taking the determinant of the both sides

$$\det(\mathfrak{w}) = \det \begin{pmatrix} 0 & \frac{2\pi\tau\mathcal{L}_0}{k_{cs}} \\ \tau & 0 \end{pmatrix} = -\frac{2\pi\tau^2\mathcal{L}_0}{k_{cs}} = \pi^2, \tag{4.37}$$

we obtain

$$\tau = i\pi\sqrt{\frac{k_{cs}}{2\pi\mathcal{L}_0}}, \quad \bar{\tau} = -i\pi\sqrt{\frac{k_{cs}}{2\pi\bar{\mathcal{L}}_0}}. \tag{4.38}$$

In the variation of the action, we consider a fixed constant modular parameter τ and $\bar{\tau}$, and, accordingly, the smoothness condition of the holonomy also fixes the constant solution \mathcal{L}_0 and $\bar{\mathcal{L}}_0$. This means that we are considering the BTZ black hole with $\tau, \bar{\tau}$ and its smooth fluctuations.

Now, we evaluate the non-constant connection $a_z(z)$, which is connected to the constant connection a_{BTZ} via a smooth residual gauge transformation by $h(z)$. In principle, using the Gauss decomposition of $h(z)$, one can find the residual gauge symmetry parameter [61]. However, within this method, it is not clear how to distinguish the large gauge transformation from the smooth one (see appendix A)

Alternatively, one may consider an infinitesimal gauge transformation, which guarantees the smoothness of gauge transformation. For example, let us consider the following (infinitesimal) gauge transformation

$$h(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda(z) + \dots, \quad \text{where} \quad \lambda(z) \equiv \begin{pmatrix} \frac{1}{2}\lambda_0(z) & -\lambda_{-1}(z) \\ \lambda_1(z) & -\frac{1}{2}\lambda_0(z) \end{pmatrix}. \tag{4.39}$$

Under the infinitesimal gauge transformation from a_{BTZ}

$$\delta a = \partial_z \lambda(z) + [a_{\text{BTZ}}, \lambda(z)], \tag{4.40}$$

one can express $\lambda_0(z)$ and $\lambda_{-1}(z)$ in terms of $\lambda_1(z)$ by demanding the transformation to keep the gauge condition in (4.31). The residual gauge transformation parametrized by $\lambda_1(z)$ leads to the transformation of the constant \mathcal{L}_0 :

$$\delta \mathcal{L} = 2\partial_z \lambda_1 \mathcal{L}_0 - \frac{k_{cs}}{4\pi} \partial_z^3 \lambda_1. \tag{4.41}$$

This is a special case of the asymptotic symmetry of AdS₃ [56, 57], or equivalently, the residual gauge symmetry of Chern-Simons gravity, which leads to Virasoro algebra. Note that from the anomaly one can determine the central charge:

$$c = 6k_{cs} = \frac{3l}{2G_N}. \tag{4.42}$$

For the conformal transformation, one can integrate the infinitesimal transformation into a finite one

$$\mathcal{L}(z) = [f'(z)]^2 \mathcal{L}_0 - \frac{k_{cs}}{4\pi} \left[\frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right], \quad (4.43)$$

where $f(z)$ is the finite residual gauge transformation (conformal transformation) parameter. Also, noting that we have

$$\mathcal{L}_0 = -\frac{\pi k_{cs}}{2\tau^2}, \quad (4.44)$$

one can write $\mathcal{L}(z)$ as

$$\begin{aligned} \mathcal{L} &= [f'(z)]^2 \mathcal{L}_0 - \frac{k_{cs}}{4\pi} \left[\frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right] \\ &= -\frac{k_{cs}}{4\pi} \text{Sch} \left[\tan \left(\frac{\pi f(z)}{\tau} \right); z \right]. \end{aligned} \quad (4.45)$$

With a similar analysis for $\bar{a}_{\bar{z}}$, the on-shell action in (4.32) is found to be

$$\begin{aligned} I_{\text{on-shell}} &= -\frac{ik_{cs}}{4\pi} \int \frac{d^2z}{\text{Im}(\tau)} \left[\tau \left(\frac{2\pi^2}{\tau^2} [f'(z)]^2 + \text{Sch}[f(z), z] \right) \right. \\ &\quad \left. - \bar{\tau} \left(\frac{2\pi^2}{\bar{\tau}^2} [\bar{f}'(\bar{z})]^2 + \text{Sch}[\bar{f}(\bar{z}), \bar{z}] \right) \right], \end{aligned} \quad (4.46)$$

which is analogous to the finite temperature Schwarzian action derived in [17].

In the metric-like formulation of 3-dimensional gravity, one implements Dirichlet boundary conditions with the Gibbons-Hawking term, which leads to interaction between the holomorphic and anti-holomorphic soft modes. This interaction, however, is suppressed at large- N (or at large central charges). On the other hand, in the Chern-Simons formalism, we did not include the Gibbons-Hawking term, and we have the decoupled on-shell action.

4.3 Four point function in Euclidean correlators

To evaluate the out-of-time-ordered correlator via analytic continuation, we start with the Euclidean four point function on the boundary. For simplicity, we consider cases where the four point function can be viewed as two point function of bi-local operators $\Phi(z_1, \bar{z}_1; z_2, \bar{z}_2)$ of which the leading contribution is one-point function (classical solution) of each bi-local operator

$$\langle \Phi_1(z_1, \bar{z}_1; z_2, \bar{z}_2) \Phi_2(z_3, \bar{z}_3; z_4, \bar{z}_4) \rangle = G_1(z_{12}, \bar{z}_{12}) G_2(z_{34}, \bar{z}_{34}) + \dots, \quad (4.47)$$

where $G_i(z_{12}, \bar{z}_{12})$ ($i = 1, 2$) is defined by

$$G_i(z_{12}, \bar{z}_{12}) \equiv \langle \Phi_i(z_1, \bar{z}_1; z_2, \bar{z}_2) \rangle \quad (i = 1, 2). \quad (4.48)$$

We now list some cases where such a bi-local operator can be considered.

Heavy and light scalar operators with sparse spectrum: one can consider the boundary-to-boundary four point function of heavy and light matter scalar fields. Then, the leading contribution will be factorized into a product of boundary-to-boundary two point functions of light and heavy operators, respectively. And, one may take the bi-local operators as follows

$$\Phi_H \sim \mathcal{O}_H \mathcal{O}_H, \quad \Phi_L \sim \mathcal{O}_L \mathcal{O}_L. \quad (4.49)$$

Wilson line and Master field in the $sl(2)$ Vasiliev equation: the Wilson line will play a role in pure Chern-Simons gravity. In particular, there has been extensive number of works on various Wilson lines in the context of higher spin gravity [48, 62–66]. Here, we can consider the simplest (boundary-to-boundary) Wilson line as our bi-local operator

$$\Phi_{\text{Wilson}}(z_1, \bar{z}_1; z_2, \bar{z}_2) = \lim_{r \rightarrow \infty} e^{4hr} \text{tr} \left[\mathcal{P} \exp \left(- \int_{r, z_2}^{r, z_1} A \right) \mathcal{P} \exp \left(- \int_{r, \bar{z}_1}^{r, \bar{z}_2} \bar{A} \right) \right], \quad (4.50)$$

where we take the r coordinate of each end symmetrically. For constant connections a and \bar{a} , this becomes a boundary-to-boundary propagator

$$\Phi_{\text{Wilson}}(z_1, \bar{z}_1; z_2, \bar{z}_2) \sim \frac{1}{\left[\sin \frac{2\pi z_{12}}{\tau} \sin \frac{2\pi \bar{z}_{12}}{\bar{\tau}} \right]^{2h}}. \quad (4.51)$$

For $sl(N)$ Chern-Simons gravity, the conformal dimension h is negative

$$h = \bar{h} = -\frac{N-1}{2} < 0. \quad (4.52)$$

In fact, a similar object was observed in the Vasiliev equation [67]. In the $SL(N)$ version of the Vasiliev system, the equation of motion for the matter master field $C \in SL(N)$ is given by

$$dC + AC - C\bar{A} = 0. \quad (4.53)$$

For the given background connections A and \bar{A} , we have

$$C(r_1, z_1, \bar{z}_1; z_2, \bar{z}_2) = \lim_{r_2 \rightarrow \infty} e^{2hr_2} \mathcal{P} \exp \left(- \int_{r_2, z_2}^{r_1, z_1} A \right) \tilde{c}_0 \mathcal{P} \exp \left(- \int_{r_1, \bar{z}_1}^{r_2, \bar{z}_2} \bar{A} \right) \quad (4.54)$$

$$= b^{-1}(r_1) \mathcal{P} \exp \left(- \int_{z_2}^{z_1} a \right) c_0 \mathcal{P} \exp \left(- \int_{\bar{z}_1}^{\bar{z}_2} \bar{a} \right) b^{-1}(r_1), \quad (4.55)$$

where \tilde{c}_0 is a constant matrix. After taking $r_2 \rightarrow \infty$ limit, it becomes “the highest weight state” i.e.,

$$\lim_{r_2 \rightarrow \infty} b(r_2) \tilde{c}_0 b^{-1}(r_2) = c_0, \quad \text{where} \quad (c_0)_{ij} \sim \delta_{i1} \delta_{j1}. \quad (4.56)$$

It was shown [67, 68] that the trace of the master field C corresponds to the physical scalar field in the higher spin gravity. In particular, for the constant connection a and \bar{a} , it becomes a bulk-to-boundary propagator [67]. Note that the coordinates (z_2, \bar{z}_2) was chosen as the initial condition of the equation of motion, and it corresponds to the position of operator inserted on the boundary. The conformal dimension h of the master field is the same as that of Wilson line

$$h = \bar{h} = -\frac{N-1}{2} < 0. \quad (4.57)$$

This is not surprising in the higher spin AdS/CFT. In the $hs[\lambda]$ higher spin gravity which contains the infinite tower of higher spin field together with scalar field, the conformal dimension of the scalar field is given by

$$h = \frac{1}{2}(1 + \lambda), \quad (4.58)$$

where $\lambda \in [0, 1]$. In the semi-classical limit where we take analytic continuation $\lambda \rightarrow -N$, one can truncate the infinite tower of higher spin field, and the gauge sector can be described by $SL(N)$ Chern-Simons gravity. In this limit, the conformal dimension of the scalar field becomes (4.57). Like Wilson line, one can also take the boundary-to-boundary propagator from the master field \mathcal{C} as our bi-local operator:

$$\Phi_{\text{Master}}(z_1, \bar{z}_1; z_2, \bar{z}_2) = \lim_{r_1 \rightarrow \infty} e^{2hr_1} \text{tr} [C(r_1, z_1, \bar{z}_1; z_2, \bar{z}_2)]. \quad (4.59)$$

In the BTZ background the modular parameter $\tau, \bar{\tau}$ becomes purely imaginary, and the periodicity of ϕ requires the one point function of bi-local operator (which will be the boundary-to-boundary propagator) to be

$$G(z_1, z_2; \bar{z}_1, \bar{z}_2) \equiv \sum_{m \in \mathbb{Z}} \frac{1}{\left[\sin \frac{\pi(z_{12} + 2\pi m)}{\tau} \right]^{2h} \left[\sin \frac{\pi(\bar{z}_{12} + 2\pi m)}{\bar{\tau}} \right]^{2\bar{h}}}. \quad (4.60)$$

Note that for the case of Wilson line this corresponds to the summation of Wilson lines winding the non-contractible cycle of solid torus. For convenience, we define

$$G_m(z_1, z_2; \bar{z}_1, \bar{z}_2) \equiv \frac{1}{\left[\sin \frac{\pi(z_{12} + 2\pi m)}{\tau} \right]^{2h} \left[\sin \frac{\pi(\bar{z}_{12} + 2\pi m)}{\bar{\tau}} \right]^{2\bar{h}}}, \quad (4.61)$$

and the boundary-to-boundary propagator can be written as

$$G_{\text{BTZ}} = \sum_{m \in \mathbb{Z}} G_m(z_1, z_2; \bar{z}_1, \bar{z}_2). \quad (4.62)$$

For the non-constant background a and \bar{a} , our bi-local operator can be understood as a gravitationally dressed operator, and we can expand our bi-local operator $\Phi(z_1, \bar{z}_1; z_2, \bar{z}_2)$ around that in the constant background $G(z_1, z_2; \bar{z}_1, \bar{z}_2)$

$$\Phi^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2) = G(z_1, \bar{z}_1; z_2, \bar{z}_2) + \epsilon G^{(1)}(z_1, \bar{z}_1; z_2, \bar{z}_2) + \dots, \quad (4.63)$$

where ϵ is an infinitesimal expansion parameter. Recall that our non-constant solution is generated from the constant BTZ solution by an infinitesimal residual gauge transformation, which corresponds to a conformal transformation on the boundary. Hence, the expansion of the dressed operator can be understood as an infinitesimal conformal transformation of the bi-local operator on the boundary. Namely, under an infinitesimal conformal transformation

$$z \quad \Longrightarrow \quad f(z) = z + \epsilon_n e^{-\frac{2\pi i n z}{\tau}}, \quad (4.64)$$

the bi-local operator can be written as

$$\begin{aligned} \Phi^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2) &= [\partial f(z_1)]^h G(f(z_1), \bar{z}_1; f(z_2), \bar{z}_2) [\partial f(z_2)]^h \\ &= G(z_1, \bar{z}_1; z_2, \bar{z}_2) + \epsilon_n \delta_{\epsilon_n} G^{\text{dressed}}(z_1, z_2; \bar{z}_1, \bar{z}_2) + \dots, \end{aligned} \quad (4.65)$$

where $\delta_{\epsilon_n} G_m^{\text{dressed}}(z_1, z_2; \bar{z}_1, \bar{z}_2)$ is found to be

$$\frac{\delta_{\epsilon_n} G^{\text{dressed}}(z_1, z_2; \bar{z}_1, \bar{z}_2)}{G(z_1, z_2; \bar{z}_1, \bar{z}_2)} = -\frac{4ih\pi}{\tau} e^{-\frac{2\pi i n \chi}{\tau}} \left[n \cos \frac{2\pi n \sigma}{\tau} - \frac{\sin \frac{2\pi n \sigma}{\tau}}{\tan \frac{2\pi(\sigma + \pi m)}{\tau}} \right]. \quad (4.66)$$

Here, we used the center of bi-local coordinates and relative coordinates defined by

$$\chi \equiv \frac{1}{2}(z_1 + z_2), \quad \sigma \equiv \frac{1}{2}(z_1 - z_2), \quad (4.67)$$

and a similar relation for the anti-holomorphic coordinates. As in [69], one can also understand it as expansion of bi-local field to construct a bi-local conformal partial wave function. For this, one can consider correlation function of the bi-local field and ϵ_{-n} , which is conjugate to ϵ_n

$$\langle \epsilon_{-n} \Phi^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2) \rangle \sim \delta_{\epsilon_n} G(z_1, \bar{z}_1; z_2, \bar{z}_2) + \dots \quad (4.68)$$

This is nothing but the conformal Ward identity.

It is more clear to see the soft mode expansion of the gravitational dressed Wilson line (as well as the master field). The bi-local operator from the Wilson line with non-constant connection $a = h^{-1} a_{\text{BTZ}} h + h^{-1} \partial h$ can be written as follows

$$\begin{aligned} \Phi_{\text{Wilson}}^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2) \\ = \lim_{r \rightarrow \infty} e^{-4|h|r} \text{tr} \left[b^{-1} h^{-1}(z_1) e^{-(z_1 - z_2) a_{\text{BTZ}}} h(z_2) [b(r)]^2 h(z_2) e^{-(\bar{z}_2 - \bar{z}_1) \bar{a}_{\text{BTZ}}} h^{-1}(z_1) b^{-1} \right]. \end{aligned} \quad (4.69)$$

For infinitesimal residual gauge parameter $h(z)$, we expand $h(z)$ around identity

$$h(z) = \mathbb{1} + \lambda(z) + \dots \quad (4.70)$$

Recall that the infinitesimal residual gauge transformation is parametrized by $\lambda_1(z)$. Then, from the mode expansion of $\lambda_1(z)$

$$\lambda_1(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n e^{-\frac{2\pi i n z}{\tau}}, \quad (4.71)$$

we can also expand the dressed Wilson line with respect to ϵ_n 's

$$\Phi_{\text{Wilson}}^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2) = \Phi_{\text{Wilson}}(z_1, \bar{z}_1; z_2, \bar{z}_2) + \sum_{n \in \mathbb{Z}} \epsilon_n [\delta_{\epsilon_n} \Phi_{\text{Wilson}}^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2)] + \dots \quad (4.72)$$

For the BTZ background, we also consider the expansion of the image of the dressed bi-local operator $\Phi_m^{\text{dressed}}(z_1, \bar{z}_1; z_2, \bar{z}_2)$ due to the periodicity of $\phi \sim \phi + 2\pi$, and we denote it by $\tilde{f}_{n,m}$

$$\begin{aligned} \tilde{f}_{n,m}(\chi, \sigma; \bar{\chi}, \bar{\sigma}) &\equiv \delta_{\epsilon_n} \Phi_m^{\text{dressed}} \\ &= -\frac{4ih\pi}{\tau} G_m(z_1, z_2; \bar{z}_1, \bar{z}_2) e^{-\frac{2\pi i n \chi}{\tau}} \left[n \cos \frac{2\pi n \sigma}{\tau} - \frac{\sin \frac{2\pi n \sigma}{\tau}}{\tan \frac{2\pi(\sigma + \pi m)}{\tau}} \right] \end{aligned} \quad (4.73)$$

In evaluating OTOCs, a particular configuration of (z_1, z_2, z_3, z_4) simplify the four point function [10, 17, 48, 70, 71]. Namely, we will consider the configuration $(z_1, \bar{z}_1; z_2, \bar{z}_2) = (\chi - \frac{\tau}{4}, \bar{\chi} - \frac{\bar{\tau}}{4}; \chi + \frac{\tau}{4}, \bar{\chi} + \frac{\bar{\tau}}{4})$ or $(\chi, \sigma, \bar{\chi}, \bar{\sigma}) = (\chi, -\frac{\tau}{4}, \bar{\chi}, -\frac{\bar{\tau}}{4})$. For example, G_m becomes

$$G_m\left(\chi, -\frac{\tau}{4}; \bar{\chi}, -\frac{\bar{\tau}}{4}\right) = \frac{1}{\left[\cos \frac{2\pi^2 m}{\tau}\right]^{2h} \left[\cos \frac{2\pi^2 m}{\bar{\tau}}\right]^{2\bar{h}}}, \quad (4.74)$$

and $\tilde{f}_{n,m}$ can also be simplified as follows

$$\tilde{f}_{n,m}\left(\chi, -\frac{\tau}{4}; \bar{\chi}, -\frac{\bar{\tau}}{4}\right) = -\frac{4ih\pi}{\tau} \frac{e^{-\frac{2\pi i n \chi}{\tau}}}{\left[\cos \frac{2\pi^2 m}{\tau} \cos \frac{2\pi^2 m}{\bar{\tau}}\right]^{2h}} \left[n \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \tan \frac{2\pi^2 m}{\tau} \right]. \quad (4.75)$$

Note that the second term is odd in m , which will be cancelled when we sum them up over the integers m . Hence, the expansion of the dressed bi-local field becomes

$$\begin{aligned} \delta_{\epsilon_n} \sum_m \Phi_m^{\text{dressed}}\left(\chi, -\frac{\tau}{4}; \bar{\chi}, -\frac{\bar{\tau}}{4}\right) &= -\frac{4ih\pi}{\tau} e^{-\frac{2\pi i n \chi}{\tau}} n \cos \frac{n\pi}{2} \sum_{m \in \mathbb{Z}} \frac{1}{\left[\cos \frac{2\pi^2 m}{\tau}\right]^{2h} \left[\cos \frac{2\pi^2 m}{\bar{\tau}}\right]^{2\bar{h}}} \\ &= -\frac{4ih\pi}{\tau} e^{-\frac{2\pi i n \chi}{\tau}} n \cos \frac{n\pi}{2} G_{\text{BTZ}}\left(\chi, -\frac{\tau}{4}; \chi, -\frac{\bar{\tau}}{4}\right). \end{aligned} \quad (4.76)$$

Note that a special choice of the configuration allow us to factor out $G_{\text{BTZ}}(\chi, -\frac{\tau}{4}; \chi, -\frac{\bar{\tau}}{4})$. In general, this is not true, in particular, the same choice of coordinates does not simplify $\delta_{\epsilon_n} \Phi_m^{\text{dressed}}$ for the case of higher spin gravity [48]. Nevertheless, each term in the summation over m has the same exponential growth in time. Hence, the summation would, at most, change the overall factor of the exponential growth, and therefore, could change the scrambling time.

Now, using the expansion of the dressed bi-local operator, we will evaluate the two point function of bi-local operators, which corresponds to the four point function of the local field. For this, we need to evaluate the correlation function of soft graviton modes, ϵ_n 's. This can be evaluated from the on-shell action (4.46) of the soft graviton mode around the constant background. Since the on-shell action is non-linear, we evaluate the quadratic action by taking an infinitesimal fluctuation around the identity

$$f(z) = z + \epsilon_n e^{-\frac{2\pi i n z}{\tau}}, \quad \bar{f}(\bar{z}) = \bar{z} + \bar{\epsilon}_n e^{-\frac{2\pi i n \bar{z}}{\tau}}. \quad (4.77)$$

Expanding the on-shell action, the quadratic action takes the form

$$I_{\text{on-shell}}^{(2)} = ik_{cs} \sum_{n \geq 2} \left[\frac{8\pi^4}{\tau^3} n^2 (n^2 - 1) \epsilon_n \epsilon_{-n} - \frac{8\pi^4}{\bar{\tau}^3} \bar{n}^2 (\bar{n}^2 - 1) \bar{\epsilon}_n \bar{\epsilon}_{-n} \right]. \quad (4.78)$$

Note that the on-shell action vanishes for $n = 0, \pm 1$, which corresponds to the isometry of constant solutions.⁹ From the quadratic on-shell action, one can read off the two point function of the boundary graviton fluctuation

$$\langle \epsilon_n \epsilon_{-n} \rangle = \frac{\kappa}{n^2 (n^2 - 1)}, \quad \langle \bar{\epsilon}_n \bar{\epsilon}_{-n} \rangle = -\frac{\bar{\kappa}}{n^2 (n^2 - 1)}, \quad (4.79)$$

where κ and $\bar{\kappa}$ are defined by

$$\kappa = \frac{\tau^3}{8\pi^4 ik_{cs}}, \quad \bar{\kappa} \equiv -\frac{\bar{\tau}^5}{8\pi^4 ik_{cs}}. \quad (4.80)$$

We will evaluate the following configuration of the OTOC regularized by the thermal density matrix $\rho \equiv e^{-\beta H}$:

$$F(t, \varphi) = \text{tr} \left[\rho^{\frac{1}{4}} V(0) \rho^{\frac{1}{4}} W(t, \varphi) \rho^{\frac{1}{4}} V(0) \rho^{\frac{1}{4}} W(t, \varphi) \right]. \quad (4.81)$$

For this, we consider the Euclidean four point function at

$$(z_1, \bar{z}_1) = \left(\chi - \frac{\tau}{4}, \bar{\chi} - \frac{\bar{\tau}}{4} \right), \quad (4.82)$$

$$(z_2, \bar{z}_2) = \left(\chi + \frac{\tau}{4}, \bar{\chi} + \frac{\bar{\tau}}{4} \right), \quad (4.83)$$

$$(z_3, \bar{z}_3) = (0, 0), \quad (4.84)$$

$$(z_4, \bar{z}_4) = \left(\frac{\tau}{2}, \frac{\bar{\tau}}{2} \right). \quad (4.85)$$

Recalling the expansion of the dressed bi-local operator in (4.76) at the special configuration, one can further simplify that of $(z_3, \bar{z}_3; z_4, \bar{z}_4)$

$$\frac{\delta G_{\text{BTZ}}(\frac{\tau}{4}, -\frac{\tau}{4}; \frac{\bar{\tau}}{4}, -\frac{\bar{\tau}}{4})}{G_{\text{BTZ}}(\frac{\tau}{4}, -\frac{\tau}{4}; \frac{\bar{\tau}}{4}, -\frac{\bar{\tau}}{4})} = -\frac{4ih\pi}{\tau} (-1)^{\frac{n}{2}} n \cos \frac{n\pi}{2}. \quad (4.86)$$

Finally, the contribution of the boundary graviton fluctuation to the Euclidean four point function can be evaluated as follows¹⁰

$$\begin{aligned} & \frac{F(1, 2, 3, 4)}{G_{\text{BTZ}}(1, 2) G_{\text{BTZ}}(3, 4)} \\ &= 1 + \sum_{|n| \geq 2} (-1)^{\frac{n}{2}} n^2 \cos^2 \frac{n\pi}{2} \left[\frac{16h^2 \pi^2}{\tau^2} e^{-\frac{2\pi i n \chi}{\tau}} \langle \epsilon_n \epsilon_{-n} \rangle + \frac{16\bar{h}^2 \pi^2}{\bar{\tau}^2} e^{-\frac{2\pi i n \bar{\chi}}{\bar{\tau}}} \langle \bar{\epsilon}_n \bar{\epsilon}_{-n} \rangle \right] \\ &= 1 + \sum_{|n|=2,4,6,\dots} \left[\frac{16h^2 \pi^2}{\tau^2} \kappa \frac{(-1)^{\frac{n}{2}} e^{-\frac{2\pi i n \chi}{\tau}}}{n^2 - 1} + \frac{16\bar{h}^2 \pi^2}{\bar{\tau}^2} \bar{\kappa} \frac{(-1)^{\frac{n}{2}} e^{-\frac{2\pi i n \bar{\chi}}{\bar{\tau}}}}{n^2 - 1} \right], \quad (4.87) \end{aligned}$$

⁹For the BTZ black hole, the $\text{SL}(2, \mathbb{C})$ isometry of AdS vacuum is broken to $\text{U}(1)$, due to the periodicity of ϕ . However, since we consider the covering space of φ , we still have the $\text{SL}(2, \mathbb{C})$ isometry. On the other hand, the soft mode eigenfunction in the BTZ background reflects this symmetry breaking, i.e., $\tilde{f}_{\pm 1, m} \neq 0$ for $m \neq 0$.

¹⁰Here, and in the following, we use the number 1 to denote (z_1, \bar{z}_1) , the number 2 to denote (z_2, \bar{z}_2) , and so on.

where only the even terms give a contribution to the four point function due to the $\cos \frac{n\pi}{2}$ factor. Then, one can rewrite the summation as a contour integral where the contour \mathcal{C} is a collection of circles centered at $n = 2\mathbb{Z}/\{0\}$ with small radius.

$$\begin{aligned} \frac{F(1, 2, 3, 4)}{G_{\text{BTZ}}(1, 2)G_{\text{BTZ}}(3, 4)} &= 1 + \frac{16h^2\pi^2}{\tau^2} \kappa \frac{1}{2\pi i} \oint_{\mathcal{C}} d\zeta \frac{\frac{\pi}{2}}{\sin \frac{\pi\zeta}{2}} \frac{e^{-\frac{2\pi i\zeta\chi}{\tau}}}{\zeta^2 - 1} \\ &+ \frac{16\bar{h}^2\pi^2}{\bar{\tau}^2} \bar{\kappa} \frac{1}{2\pi i} \oint_{\mathcal{C}} d\zeta \frac{\frac{\pi}{2}}{\sin \frac{\pi\zeta}{2}} \frac{e^{-\frac{2\pi i\zeta\bar{\chi}}{\bar{\tau}}}}{\zeta^2 - 1}. \end{aligned} \quad (4.88)$$

By pushing the contour to infinity, we pick up the simple pole at $\zeta = -1, 0, 1$, and the contour integral along \mathcal{C} becomes the residue at $\zeta = -1, 0, 1$

$$\begin{aligned} \frac{F(1, 2, 3, 4)}{G_{\text{BTZ}}(1, 2)G_{\text{BTZ}}(3, 4)} &= 1 - \frac{16h^2\pi^2}{\tau^2} \kappa \left[\frac{\pi}{4} e^{-\frac{2\pi i\chi}{\tau}} - 1 + \frac{\pi}{4} e^{\frac{2\pi i\chi}{\tau}} \right] \\ &- \frac{16\bar{h}^2\pi^2}{\bar{\tau}^2} \bar{\kappa} \left[\frac{\pi}{4} e^{-\frac{2\pi i\bar{\chi}}{\bar{\tau}}} - 1 + \frac{\pi}{4} e^{\frac{2\pi i\bar{\chi}}{\bar{\tau}}} \right]. \end{aligned} \quad (4.89)$$

4.4 Boyer-Lindquist co-rotating frame

In this section, we will shortly discuss the *Boyer-Lindquist co-rotating frame*, which is necessary for a consistent analytic continuation from Euclidean time to Lorentzian time in the four point functions.

In [72], the TFD state is interpreted as Hartle-Hawking state generated by an Euclidean time translation of size $\beta/2$. Recall that the TFD state in (2.24) can be viewed as a state generated by an $\beta/2$ Euclidean time evolution with the new twisted hamiltonians \mathcal{H}_L and \mathcal{H}_R

$$\tilde{H}_{L/R} \equiv H_{L/R} + \Omega \mathcal{J}_{L/R}, \quad (4.90)$$

where $H_{L/R}$ and $\mathcal{J}_{L/R}$ is the hamiltonian and angular momentum operator, respectively. The form of TFD state in (2.24) implies that the Hartle-Hawking state will be constructed by an Euclidean time evolution of size $\beta/2$ with the Killing vector $\tilde{H} = \partial_t + \frac{1}{\Omega} \partial_\phi$.

Note that the Killing vector \tilde{H} becomes null at horizon (the tip in the Euclidean BTZ black holes), which leads to a smooth tip of the Euclidean rotating BTZ black hole (and smooth bifurcation surface in Lorentzian geometry). Had we chosen other Killing vector to construct the Hartle-Hawking state, for example, \tilde{H} , we would not have a smooth bifurcation surface. In this sense, the twisted Hamiltonian \tilde{H} is more natural to construct the Hartle-Hawking state of the rotating BTZ black hole (assuming that this Hartle-Hawking state exists).

After constructing the Hartle-Hawking state, one can evolve the state in Lorentzian time either by H or by \tilde{H} , and they lead to different interpretation in the TFD formulation.

If (to obtain Lorentzian time correlators) we perform the analytic continuation of the Euclidean correlators, which are evolved by twisted Hamiltonian, the resulting correlator will agree with the case where the whole complex time contour (e.g., TFD time contour or Schwinger-Keldysh time contour) is evolved by \tilde{H}_L or \tilde{H}_R . Hence, we define a new time

coordinate \tilde{t} generated by \tilde{H}_L and a new angular coordinate $\tilde{\phi}$ orthogonal to the Killing vector \mathcal{H}_L

$$\tilde{t}_E \equiv \frac{1}{2} \left(t_E + \frac{1}{\Omega} i\varphi \right), \quad \tilde{\phi} \equiv \frac{1}{2} (\varphi + \Omega i t_E), \quad (4.91)$$

or

$$\tilde{t} \equiv \frac{1}{2} \left(t + \frac{1}{\Omega} \varphi \right), \quad \tilde{\phi} \equiv \frac{1}{2} (\varphi - \Omega t). \quad (4.92)$$

In our TFD formulation, the operators are defined on the space $(\tilde{t}, \tilde{\phi})$, where \tilde{t} denotes the complex time coordinates. Then, the periodicity condition becomes

$$(\tilde{t}_E, \tilde{\phi}) \sim (\tilde{t}_E - i\beta, \tilde{\phi}). \quad (4.93)$$

4.5 OTOCs from analytic continuation

From the Euclidean four point function (4.89), we consider the following term which will grow exponentially after the analytic continuation to real time

$$\begin{aligned} \frac{F(1, 2, 3, 4)}{G_{\text{BTZ}}(1, 2)G_{\text{BTZ}}(3, 4)} &= 1 - \frac{lh^2}{(r_+ + r_-)k_{cs}} \exp\left(-\frac{2\pi i}{\tau} z\right) \\ &\quad - \frac{lh^2}{(r_+ - r_-)k_{cs}} \exp\left(-\frac{2\pi i}{\bar{\tau}} \bar{z}\right) + \dots \end{aligned} \quad (4.94)$$

Here, we used

$$-\frac{4h^2\pi^3}{\tau^2} \kappa = -\frac{lh^2}{(r_+ + r_-)k_{cs}}, \quad -\frac{4\bar{h}^2\pi^3}{\bar{\tau}^2} \bar{\kappa} = -\frac{l\bar{h}^2}{(r_+ - r_-)k_{cs}}. \quad (4.95)$$

As we discussed, it is natural to use the ‘‘Boyer-Lindquist co-rotating’’ frame for the analytic continuation to the real time because the Lorentzian time is generated by the twisted Hamiltonian. In term of Boyer-Lindquist co-rotating coordinates, the holomorphic and anti-holomorphic coordinates z, \bar{z} are written as follows

$$\begin{aligned} z &= \varphi + i\frac{t_E}{l} = \frac{r_+ + r_-}{r_-} \tilde{\phi} + i\frac{r_+ - r_-}{r_+} \frac{t_E}{l}, \\ \bar{z} &= \varphi - i\frac{t_E}{l} = -\frac{r_+ - r_-}{r_-} \tilde{\phi} - i\frac{r_+ + r_-}{r_+} \frac{t_E}{l}. \end{aligned} \quad (4.96)$$

Therefore, we perform the analytic continuation of the Euclidean four point function in (4.94) within the Boyer-Lindquist co-rotating frame

$$\begin{aligned} \frac{F(1, 2, 3, 4)}{G_{\text{BTZ}}(1, 2)G_{\text{BTZ}}(3, 4)} &= 1 - \frac{2lh^2}{\pi^2(r_+ + r_-)k_{cs}} \exp\left[-\frac{(r_+ + r_-)^2}{lr_-} \tilde{\phi} - \frac{r_+^2 - r_-^2}{l^2 r_+} i\tilde{t}_E\right] \\ &\quad - \frac{2lh^2}{\pi^2(r_+ - r_-)k_{cs}} \exp\left[-\frac{(r_+ - r_-)^2}{lr_-} \tilde{\phi} - \frac{r_+^2 - r_-^2}{l^2 r_+} i\tilde{t}_E\right] \\ \implies &1 - \frac{2lh^2}{\pi^2(r_+ + r_-)k_{cs}} \exp\left[-\frac{(r_+ + r_-)^2}{lr_-} \tilde{\phi} + \frac{r_+^2 - r_-^2}{l^2 r_+} \tilde{t}\right] \\ &\quad - \frac{2lh^2}{\pi^2(r_+ - r_-)k_{cs}} \exp\left[-\frac{(r_+ - r_-)^2}{lr_-} \tilde{\phi} + \frac{r_+^2 - r_-^2}{l^2 r_+} \tilde{t}\right]. \end{aligned} \quad (4.97)$$

Hence, one can read off the Lyapunov exponent and the butterfly velocities in the Boyer-Lindquist co-rotating frame

$$\lambda_L = \frac{2\pi}{\beta}, \tag{4.98}$$

$$v_B^+ = \frac{r_-}{lr_+} \frac{r_+ - r_-}{r_+ + r_-} = \Omega \frac{1 - \ell\Omega}{1 + \ell\Omega}, \tag{4.99}$$

$$v_B^- = \frac{r_-}{lr_+} \frac{r_+ + r_-}{r_+ - r_-} = \Omega \frac{1 + \ell\Omega}{1 - \ell\Omega}. \tag{4.100}$$

In terms of the Schwarzschild coordinates, the long time behavior of the OTOC can be expressed as

$$\begin{aligned} \frac{F(1, 2, 3, 4)}{G_{\text{BTZ}}(1, 2)G_{\text{BTZ}}(3, 4)} &= 1 - \frac{2lh^2}{\pi^2(r_+ + r_-)k_{cs}} \exp\left[\frac{r_+ + r_-}{l^2}(t - l\varphi)\right] \\ &\quad - \frac{2lh^2}{\pi^2(r_+ - r_-)k_{cs}} \exp\left[\frac{r_+ - r_-}{l^2}(t + l\varphi)\right]. \end{aligned} \tag{4.101}$$

Hence, in Schwarzschild coordinates, both modes have the same butterfly velocity, $v_B/c = 1$, but different Lyapunov exponents

$$\lambda_L^\pm = \frac{r_+ \pm r_-}{\ell} = \frac{2\pi}{\beta} \frac{1}{1 \mp \ell\Omega}. \tag{4.102}$$

This result perfectly matches the result obtained with the gravity eikonal approximation in section 3.

5 Discussion

In this paper we have studied the onset of chaos in rotating BTZ black holes. We have computed OTOCs using two different methods: the elastic eikonal gravity approximation [9], and a new method that is based on the Chern-Simons formulation of 3-dimensional gravity. In the first case the OTOC can be obtained from a high energy shock wave collision in the bulk, while in the second method one explicitly derives an effective Schwarzian-like action for the boundary degrees of freedom, and computes the OTOC from analytic continuation of the Euclidean 4-point function. Both methods give the same result, which, in terms of Schwarzschild coordinates, takes the form

$$\text{OTOC}(t, \varphi_{12}) \approx 1 + C_1 e^{\frac{2\pi}{\beta_+}(t + l\varphi_{12})} + C_2 e^{\frac{2\pi}{\beta_-}(t - l\varphi_{12})}, \quad \beta_\pm = \beta(1 \mp \ell\Omega), \tag{5.1}$$

with $\ell\Omega = \frac{r_-}{r_+}$ being the chemical potential for angular momentum. The two terms correspond to left and right moving modes, which have the same butterfly velocity,¹¹ $v_B/c = 1$, but different Lyapunov exponents

$$\lambda_L^\pm = \frac{2\pi}{\beta_\pm} = \frac{2\pi}{\beta} \frac{1}{1 \mp \ell\Omega}. \tag{5.2}$$

¹¹Here $c = 1/\ell$ is the boundary speed of light.

This result is consistent with [27], where the author used the metric formulation of 3-dimensional Euclidean gravity to derive an effective action for the boundary degrees of freedom of the rotating BTZ black hole. A somewhat similar result was also found in [28], where the author studied the disruption of the mutual information in TFD states. However, due the special configuration considered in [28], their result is only sensitive to the mode with the lowest Lyapunov exponent ($\lambda_L = \frac{2\pi}{\beta_-}$).

Naively, since $\lambda_L^- \leq \frac{2\pi}{\beta} \leq \lambda_L^+$, the above result seems to indicate that one of the Lyapunov exponents is less than maximal, while the one violates the chaos bound. However, in the decomposition described in section 2.2, the parameters β_{\pm}^{-1} are precisely the effective temperatures of the left and right moving modes. Therefore, the chaos bound is different for each mode, and our result shows that each mode saturates its own chaos bound. This interpretation is very natural if one views the BTZ black hole as a part of a D1-D5 brane system. From this perspective, $\beta(1 \mp \ell\Omega)$ are the inverse temperatures of the left and right moving excitations on the effective string [73].

Co-rotating frame. The black hole temperature, β^{-1} , is defined in co-rotating coordinates (t, ϕ) , in terms of which the result takes the form

$$\text{OTOC}(t, \phi_{12}) \approx 1 + C_1 e^{\frac{2\pi}{\beta} \left(t + \frac{\ell\phi_{12}}{1 - \ell\Omega}\right)} + C_2 e^{\frac{2\pi}{\beta} \left(t - \frac{\ell\phi_{12}}{1 + \ell\Omega}\right)}, \quad (5.3)$$

In this frame, both modes have the same temperature, which results in the same Lyapunov exponent, namely $\lambda_L = \frac{2\pi}{\beta}$. This is consistent with [29], in which the authors (using the co-rotating frame) studied chaos in the rotating BTZ black hole using the geodesic approximation. Since the authors of [29] only consider homogeneous shock waves, their result has no angular dependence, and it is controlled by a single mode, with maximal Lyapunov exponent.

The change of frame breaks the symmetry between the butterfly velocities, which now become

$$\frac{v_B^{\pm}}{c} = 1 \pm \ell\Omega. \quad (5.4)$$

The rotation affects the butterfly velocities, and one of them becomes superluminal when $\Omega > 0$. Since v_B defines an effective light cone for the butterfly effect [8], one expects this quantity to be bounded by the speed of light. In fact, in asymptotically AdS geometries satisfying the Null Energy Condition (NEC) one can show that [74]

$$v_B \leq c. \quad (5.5)$$

This bound is known to be violated when the boundary theory displays non-local effects [75], which are related to a non-asymptotically AdS geometry in the dual gravitational description. Since the rotating BTZ geometry is asymptotically AdS and satisfies NEC, the violation of the bound (5.5) in this setup must have a different reason. We think this violation occurs because the rotation breaks the Z_2 isometry of the non-rotating geometry, and this introduces an asymmetry between the left and right moving modes in the co-rotating frame. This is reminiscent of the cases studied in [76–78], where the breaking

of rotational symmetry caused the violation of a different (and more strong) bound for the butterfly velocity.

Finally, we note that, at $r_- = 0$, we should recover the Rindler AdS₃ result of S&S [9] by taking the limit $\beta/\ell \ll 1$ and $\phi \ll 1$ of our results.¹² Indeed, using that¹³

$$\begin{aligned} C_1^{-1} &\propto \frac{2r_+}{\ell} \left(e^{2\pi(r_++r_-)/\ell} - 1 \right), \\ C_2^{-1} &\propto \frac{2r_+}{\ell} \left(1 - e^{-2\pi(r_+-r_-)/\ell} \right), \end{aligned} \tag{5.6}$$

one can easily check that, at $r_- = 0$, the OTOC becomes

$$\text{OTOC}(t, \phi) \propto \frac{\ell}{2r_+} \frac{1}{1 - e^{-2\pi r_+/\ell}} \left(e^{-\frac{r_+}{\ell}\phi} + e^{-\frac{2\pi r_+}{\ell}} e^{\frac{r_+}{\ell}\phi} \right), \tag{5.7}$$

where the first term matches the S&S result, while the second term ($\sim e^{\frac{r_+}{\ell}\phi}$) is exponentially suppressed¹⁴ in the limit $\beta/\ell \ll 1$ and $\phi \ll 1$.

Boyer-Lindquist co-rotating frame. In the Boyer-Lindquist coordinates, the OTOC takes the form

$$\text{OTOC}(\tilde{t}, \tilde{\varphi}_{12}) = 1 + B_1 \exp \left[\frac{2\pi}{\beta} \left(\tilde{t} - \frac{\tilde{\varphi}_{12}}{v_B^-} \right) \right] + B_2 \exp \left[\frac{2\pi}{\beta} \left(\tilde{t} + \frac{\tilde{\varphi}_{12}}{v_B^+} \right) \right], \tag{5.8}$$

where the butterfly velocities are given by

$$v_B^+ = \Omega \frac{1 - l\Omega}{1 + l\Omega}, \quad v_B^- = \Omega \frac{1 + l\Omega}{1 - l\Omega}. \tag{5.9}$$

In this frame both modes have the same, maximal, Lyapunov exponent, but they have different butterfly velocities. Just like in the case of co-rotating coordinates, one of the butterfly velocities can become superluminal, i.e., $v_B > c$. But here this only happens when $\Omega > \sqrt{2} - 1$, while in the co-rotating coordinates this happens for any value of Ω .

Extremal limit. The extremal limit occurs when $r_- \rightarrow r_+$. In the co-rotating frame (or in the Boyer-Lindquist co-rotating frame), both modes have zero Lyapunov exponent. That is consistent with the idea that the co-rotating observer only sees the zero-temperature black hole. Interestingly, this is not the case in the non-rotating frame. In this case, $\lambda_L^- = 0$, but $\lambda_L^+ = r_+/\ell^2$. That means that one of the modes survives in the extremal limit, with effective temperature given by $T = \frac{r_+}{2\pi\ell^2}$. A similar phenomenon was also observed in [79].

Acknowledgments

It is a pleasure to thank Ioannis Papadimitriou, Miok Park, Nakwoo Kim, and Juan Pedraza for extensive discussions, and Rohan R. Poojary and for useful correspondence. We also

¹²We thank Mark Mezei for calling our attention to this.

¹³See (4.6) of [38].

¹⁴Note that $\beta/\ell = 2\pi\ell/r_+ \ll 1$ implies $r_+/\ell \gg 1$. So, the second term is small because of the multiplicative factor $e^{-\frac{2\pi r_+}{\ell}}$. The first term, $e^{-\frac{r_+}{\ell}\phi}$, is not small because $\phi \ll 1$.

thank A. Misobuchi for helpful discussions about the rotating BTZ black hole, and Yongjun Ahn for carefully reviewing the section on the Eikonal approximation. The work of VJ and KK was supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (NRF2017R1A2B4004810) and GIST Research Institute(GRI) grant funded by the GIST in 2019. JY thanks the Erwin Schrodinger International Institute (ESI) where this work was completed during the program ‘‘Higher Spins and Holography 2019’’, and JY would like to thank the organizers for giving an opportunity to present our work.

A Finite residual gauge transformations

In this appendix, we will consider a finite residual gauge transformation for the constant connection a_{BTZ} . An analogous calculation holds for \bar{a}_{BTZ} . Let us start with

$$a_z = L_1 - \frac{2\pi}{k_{cs}} \mathcal{L}_0 L_{-1} = \begin{pmatrix} 0 & \frac{2\pi}{k_{cs}} \mathcal{L}_0 \\ 1 & 0 \end{pmatrix}. \quad (\text{A.1})$$

Under the finite gauge transformation by h , we have

$$a = h^{-1}(z)(d + a_{\text{BTZ}})h(z), \quad \text{where } h \in \text{SL}(2). \quad (\text{A.2})$$

By the Gauss decomposition of $h(z)$

$$h(z) = e^{h_1(z)L_1} e^{h_0(z)L_0} e^{h_{-1}(z)L_{-1}} = \begin{pmatrix} 1 & 0 \\ h_1(z) & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{h_0(z)}{2}} & 0 \\ 0 & e^{-\frac{h_0(z)}{2}} \end{pmatrix} \begin{pmatrix} 1 & -h_{-1}(z) \\ 0 & 0 \end{pmatrix}, \quad (\text{A.3})$$

one can express h_0 and h_{-1} in terms of h_1 by demanding that the gauge transformation by $h(z)$ keeps the gauge condition in (4.31):

$$a = L_1 - \frac{2\pi}{k_{cs}} \mathcal{L} L_{-1} = \begin{pmatrix} 0 & \frac{2\pi}{k_{cs}} \mathcal{L}(z) \\ 1 & 0 \end{pmatrix}. \quad (\text{A.4})$$

We obtain

$$e^{-h_0(z)} = 1 - \frac{2\pi}{k_{cs}} \mathcal{L}_0 [h_1(z)]^2 + h_1'(z), \quad (\text{A.5})$$

$$h_{-1}(z) = -\frac{2\pi}{k_{cs}} \mathcal{L}_0 h_1(z) - \frac{1}{2} h_0(z)', \quad (\text{A.6})$$

and $\mathcal{L}[h_1]$ is a complicated functional of h_1 . It is better to reparametrize $h_1(z)$ by defining $\phi(z)$ via

$$h_1(z) = -\frac{1}{\sqrt{-\frac{2\pi}{k_{cs}} \mathcal{L}_0}} \tan \left[\sqrt{-\frac{2\pi}{k_{cs}} \mathcal{L}_0} (t - \phi(z)) \right]. \quad (\text{A.7})$$

In this case $\mathcal{L}(z)$ becomes the finite temperature Schwarzian

$$\mathcal{L}(z) = -\frac{k_{cs}}{4\pi} \left[\frac{\phi'''}{\phi'} - \frac{3}{2} \left(\frac{\phi''}{\phi'} \right)^2 \right] + \mathcal{L}_0 (\phi')^2. \quad (\text{A.8})$$

From (4.44), the gauge parameter $h(z)$ can be obtained to be

$$h(z) = \begin{pmatrix} \cos\left(\frac{\pi}{\tau}(z - \phi(z))\right) & \frac{\pi}{\tau} \sin\left(\frac{\pi}{\tau}(z - \phi(z))\right) \\ -\frac{\sin\left(\frac{\pi}{\tau}(z - \phi(z))\right)}{\frac{\pi}{\tau}} & \cos\left(\frac{\pi}{\tau}(z - \phi(z))\right) \end{pmatrix} \begin{pmatrix} \frac{1}{\phi'(z)^{\frac{1}{2}}} & -\frac{\phi''(z)}{2\phi'(z)^{\frac{3}{2}}} \\ 0 & \phi'(z)^{\frac{1}{2}} \end{pmatrix}. \quad (\text{A.9})$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [[hep-th/9711200](#)] [[INSPIRE](#)].
- [2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [4] G. Sárosi, *AdS₂ holography and the SYK model*, *PoS(Modave2017)001* [[arXiv:1711.08482](#)] [[INSPIRE](#)].
- [5] V. Jahnke, *Recent developments in the holographic description of quantum chaos*, *Adv. High Energy Phys.* **2019** (2019) 9632708 [[arXiv:1811.06949](#)] [[INSPIRE](#)].
- [6] S.H. Shenker and D. Stanford, *Black holes and the butterfly effect*, *JHEP* **03** (2014) 067 [[arXiv:1306.0622](#)] [[INSPIRE](#)].
- [7] S.H. Shenker and D. Stanford, *Multiple shocks*, *JHEP* **12** (2014) 046 [[arXiv:1312.3296](#)] [[INSPIRE](#)].
- [8] D.A. Roberts, D. Stanford and L. Susskind, *Localized shocks*, *JHEP* **03** (2015) 051 [[arXiv:1409.8180](#)] [[INSPIRE](#)].
- [9] S.H. Shenker and D. Stanford, *Stringy effects in scrambling*, *JHEP* **05** (2015) 132 [[arXiv:1412.6087](#)] [[INSPIRE](#)].
- [10] J. Maldacena, S.H. Shenker and D. Stanford, *A bound on chaos*, *JHEP* **08** (2016) 106 [[arXiv:1503.01409](#)] [[INSPIRE](#)].
- [11] J. de Boer, E. Lladrés, J.F. Pedraza and D. Vegh, *Chaotic strings in AdS/CFT*, *Phys. Rev. Lett.* **120** (2018) 201604 [[arXiv:1709.01052](#)] [[INSPIRE](#)].
- [12] A. Banerjee, A. Kundu and R.R. Poojary, *Strings, branes, Schwarzian action and maximal chaos*, [arXiv:1809.02090](#) [[INSPIRE](#)].
- [13] A. Banerjee, A. Kundu and R. Poojary, *Maximal chaos from strings, branes and Schwarzian action*, [arXiv:1811.04977](#) [[INSPIRE](#)].
- [14] S. Sachdev and J. Ye, *Gapless spin fluid ground state in a random, quantum Heisenberg magnet*, *Phys. Rev. Lett.* **70** (1993) 3339 [[cond-mat/9212030](#)] [[INSPIRE](#)].
- [15] J. Polchinski and V. Rosenhaus, *The spectrum in the Sachdev-Ye-Kitaev model*, *JHEP* **04** (2016) 001 [[arXiv:1601.06768](#)] [[INSPIRE](#)].
- [16] A. Jevicki, K. Suzuki and J. Yoon, *Bi-local holography in the SYK model*, *JHEP* **07** (2016) 007 [[arXiv:1603.06246](#)] [[INSPIRE](#)].
- [17] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** (2016) 106002 [[arXiv:1604.07818](#)] [[INSPIRE](#)].

- [18] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [[arXiv:1606.01857](#)] [[INSPIRE](#)].
- [19] A. Jevicki and K. Suzuki, *Bi-local holography in the SYK model: perturbations*, *JHEP* **11** (2016) 046 [[arXiv:1608.07567](#)] [[INSPIRE](#)].
- [20] D.J. Gross and V. Rosenhaus, *A generalization of Sachdev-Ye-Kitaev*, *JHEP* **02** (2017) 093 [[arXiv:1610.01569](#)] [[INSPIRE](#)].
- [21] W. Fu, D. Gaiotto, J. Maldacena and S. Sachdev, *Supersymmetric Sachdev-Ye-Kitaev models*, *Phys. Rev. D* **95** (2017) 026009 [[arXiv:1610.08917](#)] [[INSPIRE](#)].
- [22] C. Ahn and C. Peng, *Chiral algebras of two-dimensional SYK models*, [arXiv:1812.05106](#) [[INSPIRE](#)].
- [23] A.M. García-García, T. Nosaka, D. Rosa and J.J.M. Verbaarschot, *Quantum chaos transition in a two-site SYK model dual to an eternal traversable wormhole*, [arXiv:1901.06031](#) [[INSPIRE](#)].
- [24] R. de Mello Koch, W. LiMing, H.J.R. Van Zyl and J.P. Rodrigues, *Chaos in the fishnet*, [arXiv:1902.06409](#) [[INSPIRE](#)].
- [25] J. Kim, I.R. Klebanov, G. Tarnopolsky and W. Zhao, *Symmetry breaking in coupled SYK or tensor models*, [arXiv:1902.02287](#) [[INSPIRE](#)].
- [26] F. Ferrari and F.I. Schaposnik Massolo, *On phases of melonic quantum mechanics*, [arXiv:1903.06633](#) [[INSPIRE](#)].
- [27] R.R. Poojary, *BTZ dynamics and chaos*, [arXiv:1812.10073](#) [[INSPIRE](#)].
- [28] A. Štikonas, *Scrambling time from local perturbations of the rotating BTZ black hole*, *JHEP* **02** (2019) 054 [[arXiv:1810.06110](#)] [[INSPIRE](#)].
- [29] A.P. Reynolds and S.F. Ross, *Butterflies with rotation and charge*, *Class. Quant. Grav.* **33** (2016) 215008 [[arXiv:1604.04099](#)] [[INSPIRE](#)].
- [30] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)] [[INSPIRE](#)].
- [31] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2 + 1) black hole*, *Phys. Rev. D* **48** (1993) 1506 [*Erratum ibid.* **D 88** (2013) 069902] [[gr-qc/9302012](#)] [[INSPIRE](#)].
- [32] V. Balasubramanian and T.S. Levi, *Beyond the veil: inner horizon instability and holography*, *Phys. Rev. D* **70** (2004) 106005 [[hep-th/0405048](#)] [[INSPIRE](#)].
- [33] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, *The black hole singularity in AdS/CFT*, *JHEP* **02** (2004) 014 [[hep-th/0306170](#)] [[INSPIRE](#)].
- [34] E. Keski-Vakkuri, *Bulk and boundary dynamics in BTZ black holes*, *Phys. Rev. D* **59** (1999) 104001 [[hep-th/9808037](#)] [[INSPIRE](#)].
- [35] P. Caputa, G. Mandal and R. Sinha, *Dynamical entanglement entropy with angular momentum and U(1) charge*, *JHEP* **11** (2013) 052 [[arXiv:1306.4974](#)] [[INSPIRE](#)].
- [36] G. 't Hooft, *Graviton dominance in ultrahigh-energy scattering*, *Phys. Lett. B* **198** (1987) 61 [[INSPIRE](#)].

- [37] D.N. Kabat and M. Ortiz, *Eikonal quantum gravity and Planckian scattering*, *Nucl. Phys. B* **388** (1992) 570 [[hep-th/9203082](#)] [[INSPIRE](#)].
- [38] Z. Fu, B. Grado-White and D. Marolf, *A perturbative perspective on self-supporting wormholes*, *Class. Quant. Grav.* **36** (2019) 045006 [[arXiv:1807.07917](#)] [[INSPIRE](#)].
- [39] K. Jensen, *Chaos in AdS₂ holography*, *Phys. Rev. Lett.* **117** (2016) 111601 [[arXiv:1605.06098](#)] [[INSPIRE](#)].
- [40] D. Bagrets, A. Altland and A. Kamenev, *Sachdev–Ye–Kitaev model as Liouville quantum mechanics*, *Nucl. Phys. B* **911** (2016) 191 [[arXiv:1607.00694](#)] [[INSPIRE](#)].
- [41] D. Grumiller et al., *Menagerie of AdS₂ boundary conditions*, *JHEP* **10** (2017) 203 [[arXiv:1708.08471](#)] [[INSPIRE](#)].
- [42] G. Mandal, P. Nayak and S.R. Wadia, *Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models*, *JHEP* **11** (2017) 046 [[arXiv:1702.04266](#)] [[INSPIRE](#)].
- [43] M. Cvetič and I. Papadimitriou, *AdS₂ holographic dictionary*, *JHEP* **12** (2016) 008 [Erratum *ibid.* **1701** (2017) 120] [[arXiv:1608.07018](#)] [[INSPIRE](#)].
- [44] S.R. Das, A. Jevicki and K. Suzuki, *Three dimensional view of the SYK/AdS duality*, *JHEP* **09** (2017) 017 [[arXiv:1704.07208](#)] [[INSPIRE](#)].
- [45] S.R. Das, A. Ghosh, A. Jevicki and K. Suzuki, *Three dimensional view of arbitrary q SYK models*, *JHEP* **02** (2018) 162 [[arXiv:1711.09839](#)] [[INSPIRE](#)].
- [46] A. Castro, F. Larsen and I. Papadimitriou, *5D rotating black holes and the nAdS₂/nCFT₁ correspondence*, *JHEP* **10** (2018) 042 [[arXiv:1807.06988](#)] [[INSPIRE](#)].
- [47] A. Gaikwad, L.K. Joshi, G. Mandal and S.R. Wadia, *Holographic dual to charged SYK from 3D gravity and Chern-Simons*, [arXiv:1802.07746](#) [[INSPIRE](#)].
- [48] P. Narayan and J. Yoon, *Chaos in three-dimensional higher spin gravity*, to appear (2019).
- [49] J. de Boer and J.I. Jottar, *Thermodynamics of higher spin black holes in AdS₃*, *JHEP* **01** (2014) 023 [[arXiv:1302.0816](#)] [[INSPIRE](#)].
- [50] A. Castro, N. Iqbal and E. Lladrés, *Eternal higher spin black holes: a thermofield interpretation*, *JHEP* **08** (2016) 022 [[arXiv:1602.09057](#)] [[INSPIRE](#)].
- [51] S. Carlip and C. Teitelboim, *Aspects of black hole quantum mechanics and thermodynamics in (2 + 1)-dimensions*, *Phys. Rev. D* **51** (1995) 622 [[gr-qc/9405070](#)] [[INSPIRE](#)].
- [52] A. Achúcarro and P.K. Townsend, *A Chern-Simons action for three-dimensional Anti-de Sitter supergravity theories*, *Phys. Lett. B* **180** (1986) 89 [[INSPIRE](#)].
- [53] E. Witten, *(2 + 1)-dimensional gravity as an exactly soluble system*, *Nucl. Phys. B* **311** (1988) 46 [[INSPIRE](#)].
- [54] K. Krasnov, *On holomorphic factorization in asymptotically AdS 3D gravity*, *Class. Quant. Grav.* **20** (2003) 4015 [[hep-th/0109198](#)] [[INSPIRE](#)].
- [55] G. Arcioni, M. Blau and M. O’Loughlin, *On the boundary dynamics of Chern-Simons gravity*, *JHEP* **01** (2003) 067 [[hep-th/0210089](#)] [[INSPIRE](#)].
- [56] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [[arXiv:1008.4744](#)] [[INSPIRE](#)].

- [57] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic W-symmetries in three-dimensional higher-spin gauge theories*, *JHEP* **09** (2011) 113 [[arXiv:1107.0290](#)] [[INSPIRE](#)].
- [58] M. Gutperle and P. Kraus, *Higher spin black holes*, *JHEP* **05** (2011) 022 [[arXiv:1103.4304](#)] [[INSPIRE](#)].
- [59] A. Castro, E. Hijano, A. Lepage-Jutier and A. Maloney, *Black holes and singularity resolution in higher spin gravity*, *JHEP* **01** (2012) 031 [[arXiv:1110.4117](#)] [[INSPIRE](#)].
- [60] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, *Conical defects in higher spin theories*, *JHEP* **02** (2012) 096 [[arXiv:1111.3381](#)] [[INSPIRE](#)].
- [61] O. Coussaert, M. Henneaux and P. van Driel, *The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant*, *Class. Quant. Grav.* **12** (1995) 2961 [[gr-qc/9506019](#)] [[INSPIRE](#)].
- [62] M. Ammon, A. Castro and N. Iqbal, *Wilson lines and entanglement entropy in higher spin gravity*, *JHEP* **10** (2013) 110 [[arXiv:1306.4338](#)] [[INSPIRE](#)].
- [63] J. de Boer and J.I. Jottar, *Entanglement entropy and higher spin holography in AdS₃*, *JHEP* **04** (2014) 089 [[arXiv:1306.4347](#)] [[INSPIRE](#)].
- [64] A. Castro, S. Detournay, N. Iqbal and E. Perlmutter, *Holographic entanglement entropy and gravitational anomalies*, *JHEP* **07** (2014) 114 [[arXiv:1405.2792](#)] [[INSPIRE](#)].
- [65] J. de Boer et al., *Higher spin entanglement and \mathcal{W}_N conformal blocks*, *JHEP* **07** (2015) 168 [[arXiv:1412.7520](#)] [[INSPIRE](#)].
- [66] A. Castro, N. Iqbal and E. Llabrés, *Wilson lines and Ishibashi states in AdS₃/CFT₂*, *JHEP* **09** (2018) 066 [[arXiv:1805.05398](#)] [[INSPIRE](#)].
- [67] E. Hijano, P. Kraus and E. Perlmutter, *Matching four-point functions in higher spin AdS₃/CFT₂*, *JHEP* **05** (2013) 163 [[arXiv:1302.6113](#)] [[INSPIRE](#)].
- [68] M. Ammon, P. Kraus and E. Perlmutter, *Scalar fields and three-point functions in D = 3 higher spin gravity*, *JHEP* **07** (2012) 113 [[arXiv:1111.3926](#)] [[INSPIRE](#)].
- [69] R. de Mello Koch, A. Jevicki, K. Suzuki and J. Yoon, *AdS maps and diagrams of bi-local holography*, *JHEP* **03** (2019) 133 [[arXiv:1810.02332](#)] [[INSPIRE](#)].
- [70] J. Yoon, *SYK models and SYK-like tensor models with global symmetry*, *JHEP* **10** (2017) 183 [[arXiv:1707.01740](#)] [[INSPIRE](#)].
- [71] P. Narayan and J. Yoon, *Supersymmetric SYK model with global symmetry*, *JHEP* **08** (2018) 159 [[arXiv:1712.02647](#)] [[INSPIRE](#)].
- [72] J.M. Maldacena, *Eternal black holes in Anti-de Sitter*, *JHEP* **04** (2003) 021 [[hep-th/0106112](#)] [[INSPIRE](#)].
- [73] J.M. Maldacena and A. Strominger, *AdS₃ black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [[hep-th/9804085](#)] [[INSPIRE](#)].
- [74] X.-L. Qi and Z. Yang, *Butterfly velocity and bulk causal structure*, [arXiv:1705.01728](#) [[INSPIRE](#)].
- [75] W. Fischler, V. Jahnke and J.F. Pedraza, *Chaos and entanglement spreading in a non-commutative gauge theory*, *JHEP* **11** (2018) 072 [[arXiv:1808.10050](#)] [[INSPIRE](#)].

- [76] V. Jahnke, *Delocalizing entanglement of anisotropic black branes*, *JHEP* **01** (2018) 102 [[arXiv:1708.07243](#)] [[INSPIRE](#)].
- [77] D. Giataganas, U. Gürsoy and J.F. Pedraza, *Strongly-coupled anisotropic gauge theories and holography*, *Phys. Rev. Lett.* **121** (2018) 121601 [[arXiv:1708.05691](#)] [[INSPIRE](#)].
- [78] D. Avila, V. Jahnke and L. Patiño, *Chaos, diffusivity and spreading of entanglement in magnetic branes and the strengthening of the internal interaction*, *JHEP* **09** (2018) 131 [[arXiv:1805.05351](#)] [[INSPIRE](#)].
- [79] P. Caputa, V. Jejjala and H. Soltanpanahi, *Entanglement entropy of extremal BTZ black holes*, *Phys. Rev. D* **89** (2014) 046006 [[arXiv:1309.7852](#)] [[INSPIRE](#)].