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Distributed formation control of the special Euclidean group SE(2) via global orientation control

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Abstract: The authors propose fully distributed strategies for multi-agent formation control of the two-dimensional special Euclidean group. To control the rotated local reference frames, they firstly estimate the orientation angle of each agent with regard to a global reference frame. By using only local measurements, the orientation angle is estimated in a distributed way. The estimated orientation is, then, exploited to control each agent's orientation. Each agent's orientation converges to the prescribed desired orientation. Finally, a distributed formation control strategy based on displacement measurement is proposed to achieve the desired formation shape of the two-dimensional special Euclidean group. Under the proposed strategy, the authors ensure that the formation shape including position and orientation exponentially converges to the desired formation shape. Moreover, the formation shape is invariant to a translation and a rotation compared with the desired formation shape.

1 Introduction

Formation control of multi-agent systems has been extensively studied from various perspectives: time-varying formation control [1–3], control schemes [4–7] and control or sensing variables [8–15]. Several formation control strategies have been proposed by using global or local information associated with a global reference frame [1–8]. The formation control becomes more challenging in fully distributed environment using local information associated with a local reference frame. To solve this problem, the fully distributed strategies have been widely studied in the literature [9–14].

Displacement-based formation control is a typical distributed approach for a multi-agent system [8–11]. In the literature, a displacement is referred to as the relative position of two agents. Since the relative position is often considered as a vector having the direction with regard to each agent's local reference frame, the displacement-based formation problem is usually solved by assuming that all agents share a common reference frame [8]. However, it does not ensure convergence of the desired formation shape when the agents' orientations are misaligned. The orientation misalignment appears in various practical situations. For example, let us consider magnetometers that can be used for establishing local reference frames in multi-drone system [16, 17]. Since each drone senses or moves on its local reference frame measured by magnetometers, it is unrealistic to ignore the effect of misaligned local frames between drones. For the orientation misalignment problem, several solutions have been proposed [9–11]. In the work of [9], orientation alignment method is proposed to make a desired formation shape. The proposed method ensures the asymptotic convergence in the formation control problem with the uniformly connected interaction topology. Lee and Ahn proposed an orientation estimation method to solve the distributed formation control problem of a multi-agent system in the case of misaligned frames [10, 11]. However, since formation control methods via orientation estimation cannot ensure the freedom of orientations in the special orthogonal group, the works of [10, 11, 14] do not describe the formation control of the special Euclidean group.

Formation control of the special Euclidean group is controlling both orientation and formation of agents characterised in $\mathrm{SE}(d)=\mathrm{SO}(d)\times\mathbb{R}^d$. The formation control of the special Euclidean group has been studied in [18–21]. In the case of the rotated local reference frames, several methods have been proposed, such as a common Cartesian coordinate coupling [18, 19], a rotation matrix coupling [20] and a Euclidean transformation coupling [21]. The formation control of the special Euclidean is employed in various applications of multi-agent coordination, such as earth observation using synthetic aperture radars [22, 23] and pose estimation using multi-camera networks [24]. Since the main objective of these applications is to observe a common object in multi-sensor networks, sensors fixed to each agent's body frame should aim at a common object while all agents achieve a desired formation shape. In the perspective of formation control, it implies that agents are necessary to achieve not only their own desired positions but also their own desired orientations.

Several existing works for distributed formation control [8–11] did not propose the orientation control methods. Since agents always maintain their initial orientations or are aligned to a common reference frame, these works does not ensure formation control of the special Euclidean group. Moreover, the existing formation control of special Euclidean group [18–21] did not provide a solution to achieve the desired formation shape in distributed manners when agents' orientations are rotated to point towards their target orientations. In this paper, we propose a fully distributed formation control strategy considering both the relative orientation angle and position of agents in two-dimensional space. Also, by verifying that the formation shape is invariant to a translation and the rotation compared to the desired formation, we show that the proposed strategy guarantees the distributed formation control of the special Euclidean group $\mathrm{SE}(2)=\mathrm{SO}(2)\times\mathbb{R}^2$. The main contributions of this paper can be summarised as follows. First, we control agents' orientations by using dispersively estimated orientations with regard to a global reference frame. The proposed orientation control method does not use any information about the global reference frame and ensures exponential convergence to the desired orientation containing the common offset. Second, a distributed formation control strategy is proposed to achieve the desired formation shape of the two-dimensional special Euclidean group. This formation control strategy also does not use any global position information and ensures exponential convergence of the desired formation shape.

The rest of this paper is organised as follows. In Section 2, we briefly explain the mathematical background and state the main problems. The orientation control strategy using the estimated global orientations and the distributed formation control strategy are proposed and analysed in Section 3. Simulation results are provided in Section 4 and the conclusion is presented in Section 5.

2 Preliminaries and problem statement

The matrices I_m denotes the m by m identity matrix, and $1_m = [1, \dots, 1]^{\mathrm{T}} \in \mathbb{R}^m$ and $0_m = [0, \dots, 0]^{\mathrm{T}} \in \mathbb{R}^m$, where $\mathbb{R}^m$ denotes the m-dimensional Euclidean space. The operator $\otimes$ denotes the Kronecker product of the matrices. For a rotation matrix $R \in \mathrm{SO}(2)$, the determinant of R is always 1, i.e. $\det(R) = 1$. Moreover $R^{\mathrm{T}}R = I_2$.

2.1 Interaction topology

The interaction topology for a group of agents can be expressed by a directed weighted graph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ or an undirected weighted graph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the sets of vertices and edges are denoted as $\mathcal{V} = \{1, \dots, n\}$ and $\mathcal{E} = \{(i, j) \mid i, j \in \mathcal{V}, i \neq j\} \subset \mathcal{V} \times \mathcal{V}$, respectively, and a weight set is denoted as $\mathcal{A} = \{\alpha_{ij} > 0 \mid (i, j) \in \mathcal{E}\}$. Further, the set of vertex i's neighbours is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. In the directed graph $\mathcal{G}_d$, an edge (i,j) denotes that i's information is only communicated to j. However, in the undirected graph $\mathcal{G}_u$, the edge (i,j) means that j's information is also communicated to i. A spanning tree of $\mathcal{G}_u$ is an undirected graph which is a connected graph including all vertices of $\mathcal{G}_u$. An arborescence is a directed graph which is an acyclic connected graph that has a root node having one directed path to every other nodes. A rooted acyclic digraph is a specific arborescence which is constructed by the following process. A vertex 1 has no neighbour. Add a vertex 2 which has the directed edge $(2, 1)$. Add a vertex 3 which has one directed edge or two directed edges $(3, j)$, for some j's $\in \{1, 2\}$. Sequentially, we can add a new vertex $i (3 \leq i \leq n)$ and one or multiple directed edges (i,j), for some j's $\in \{1, 2, \dots, i-1\}$. The Laplacian matrix of a graph is denoted as $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ where $l_{ij} = -\alpha_{ij}$ and $l_{ij} = \sum_{j \in \mathcal{N}_i} \alpha_{ij}$ if $i = j$.

Consider an n-agent system whose interaction is modelled as a(an) directed(undirected) weighted graph $\mathcal{G}_{d(u)} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. A consensus protocol is proposed in the work of [25] as follows:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} \alpha_{ij}(x_j(t) - x_i(t)), \forall i \in \mathcal{V} \tag{1}$$

where $x_i \in \mathbb{R}^m$. The protocol can be written as follows:

$$\dot{\mathbf{x}}(t) = -(L \otimes I_m)\mathbf{x}(t) \tag{2}$$

where $\mathbf{x} = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_n^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{mn}$. From the work of [26], the equilibrium set of (2) is globally exponentially stable under the interaction topology having an arborescence [26].

2.2 Consensus of the special Euclidean group

Consider the rotation matrix in d-dimensional space. The special orthogonal group is written as follows:

$$\mathrm{SO}(d) = \{R_i \in \mathbb{R}^{d \times d} \mid R_i^{\mathrm{T}} R_i = I_d, \det(R_i) = 1\} \tag{3}$$

where R_i is a rotation matrix in $\mathbb{R}^d$. From the work of [18], the consensus protocol of (1) can be rewritten as follows:

$$\dot{p}_i(t) = \sum_{j \in \mathcal{N}_i} \alpha_{ij} R(p_j(t) - p_i(t)), \quad \forall i \in \mathcal{V} \tag{4}$$

where p_i is agent i's position and R is a common rotation matrix. The position of agents is controlled by coupling the rotation matrix. In the work of [18], the consensus algorithm of (4) ensures the collective motion of agents such as rendezvous, circular patterns and logarithmic spiral patterns. Let us consider the group of Euclidean transformations in d-dimensional space. The special Euclidean group is written as follows:

$$\mathrm{SE}(d) = \left\{ G_i = \begin{bmatrix} R_i & p_i \\ 0_d^{\mathrm{T}} & 1 \end{bmatrix} \middle| R_i \in \mathrm{SO}(d),\ p_i \in \mathbb{R}^d \right\} \tag{5}$$

where p_i is a position vector in $\mathbb{R}^d$. In the work of [21], the dynamics of agents is written as follows:

$$\dot{G}_i = G_i \sum_{j \in \mathcal{N}_i^{\sigma_i(t)}} \alpha_{ij}(G_{ij} - G_{ij}^{-1}), \quad \forall i \in \mathcal{V} \tag{6}$$

where G_i is agent i's Euclidean transformation in SE(3) and $\mathcal{N}_i^{\sigma_i(t)}$ is the agent i's time-varying neighbourhood. The control law (6) leads to consensus of agents. From [18–21], agents are characterised in the special Euclidean group $\mathrm{SE}(d) = \mathrm{SO}(d) \times \mathbb{R}^d$. By coupling or controlling both orientations and positions of agent, they solve the consensus problem of the special Euclidean group.

2.3 Problem statement

Consider a n-agent system modelled by a single integrator as follows:

$$\dot{p}_i(t) = u_i(t), \quad i \in \mathcal{V} \triangleq \{1, \dots, n\} \tag{7}$$

where $p_i(t) \in \mathbb{R}^2$ is the position of agent i and $u_i(t) \in \mathbb{R}^2$ is the control input of agent i with regard to a global reference frame, denoted by ${}^g\Sigma$. We let ${}^i\Sigma$ be a ith local reference frame. An orientation of ${}^i\Sigma$ with respect to ${}^g\Sigma$ is identified by the rotation matrix $R_i \in \mathrm{SO}(2)$. The rotation matrix R_i can be expressed in the orientation angle θ_i as follows:

$$R_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{bmatrix}.$$

Let $p_i^i(t) \in \mathbb{R}^2$ denote agent i's position with regard to the local reference frame ${}^i\Sigma$. Then, the position dynamics of (7) can be rewritten as follows:

$$\dot{p}_i^i(t) = u_i^i(t), \quad i \in \mathcal{V} \triangleq \{1, \dots, n\} \tag{8}$$

where the velocity of agent i is rotated by the local reference frame such that $\dot{p}_i^i = R_i \dot{p}_i$.

The interaction topology of the multi-agent system includes both a sensing graph and a communication graph. The sensing graph $\mathcal{G}_u^s = (\mathcal{V}, \mathcal{E}^s)$ is assumed to have bidirectional edges while the communication graph $\mathcal{G}_d^c = (\mathcal{V}, \mathcal{E}^c)$ is not necessary to be an undirected graph. Each agent can obtain the sensor data under a sensing graph and transfer the sensor data under a communication graph. In this work, the interaction topology satisfies the following Assumption 1.

Assumption 1: The sensing graph $\mathcal{G}_u^s$ has a spanning tree, and the communication graph $\mathcal{G}_d^c$ is a rooted acyclic digraph. Further, $\mathcal{G}_d^c$ is a subgraph of $\mathcal{G}_u^s$.

Under the sensing graph, each agent measures both relative position and relative orientation of its neighbours. The relative position of neighbours is expressed in the ith local reference frame as follows:

$$p_{ji}^i(t) \triangleq p_j^i(t) - p_i^i(t), \quad i \in \mathcal{V},\ j \in \mathcal{N}_i \tag{9}$$

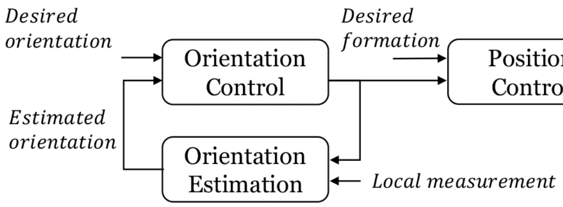

Fig. 1 *Block diagram for formation control via orientation control*

where a superscript i implies the local reference frame of ith agent. Note that $p_j^i - p_i^i = R_i(p_j - p_i)$. The relative orientation angle is defined as

$$\theta_{ij}(t) \triangleq \mathrm{PV}(\theta_i(t) - \theta_j(t)), \quad i \in \mathscr{V},\ j \in \mathscr{N}_i \tag{10}$$

where the prime value operation is defined as PV $(\theta_i - \theta_j) \triangleq [(\theta_i - \theta_j + \pi) \bmod 2\pi] - \pi$. The following equation is proved in the work of [9]:

$$\theta_{ij} \triangleq \mathrm{PV}(\theta_i - \theta_j) = \mathrm{PV}(\boldsymbol{\delta}_{ij} - \boldsymbol{\delta}_{ji} + \pi) \tag{11}$$

where $\boldsymbol{\delta}_{ij}$ and $\boldsymbol{\delta}_{ji}$ are angle values of $p_{ij}^i \in \mathbb{R}^2$ and $p_{ji}^j \in \mathbb{R}^2$, respectively [we can express $p_{ij}^i = \| p_{ij}^i \| [\cos(\boldsymbol{\delta}_{ij}), \sin(\boldsymbol{\delta}_{ij})]^{\mathrm{T}}$]. Under the communication graph, the agents should communicate p_{ij}^i and p_{ji}^j to calculate the relative orientation angle according to (11). It shows that agent i can calculate the relative orientation angle by using relative position measurements.

In this paper, by controlling agents characterised by $(R_i, p_i) \in \mathrm{SO}(2) \times \mathbb{R}^2$, we solve the formation control problem of the special Euclidean group. That is, we attempt to make a desired formation shape for both position and orientation of agents by using the relative position measurement. The relative orientation of neighbours can be obtained from the relative position measurement (see (11)). Based on the relative orientation, we estimate the orientation of each agent. The estimated orientation is exploited to control the orientation to the desired orientation. Let the desired orientation $R_i^* \in \mathrm{SO}(2)$, $i \in \mathscr{V}$ be given. The orientation control problem is then stated as follows.

Problem 1: Consider n agents in two-dimensional space. Suppose that the interaction topology includes $\mathscr{G}_u^s = (\mathscr{V}, \mathscr{E}^s)$ and $\mathscr{G}_d^c = (\mathscr{V}, \mathscr{E}^c)$ under Assumption 1. Based on the relative orientation R_{ij}, design an orientation control law such that $R_{ij}(t) = R_i(t)(R_j(t))^{\mathrm{T}} \to R_i^*(R_j^*)^{\mathrm{T}}\ \forall i, j \in \mathscr{V}$ as $t \to \infty$.

Let us consider the desired formation $\mathbf{p}^* = \{p_1^*, \ldots, p_n^*\}$. Since each agent has its own desired orientation, it is necessary to compensate the desired relative position in the view of the local reference frame. The desired relative position for ith agent can be denoted as $R_i^*(p_j^* - p_i^*)$, $j \in \mathscr{N}_i$, $i \in \mathscr{V}$. The formation control problem is, then, stated as follows:

Problem 2: Consider n agents in two-dimensional space. Suppose that the interaction topology includes $\mathscr{G}_u^s = (\mathscr{V}, \mathscr{E}^s)$ and $\mathscr{G}_d^c = (\mathscr{V}, \mathscr{E}^c)$ under Assumption 1. Design a position control law such that $(p_j^i - p_i^i) \to R_i^*(p_i^* - p_j^*)\ \forall i, j \in \mathscr{V}$ as $t \to \infty$.

The block diagram of the proposed formation control strategy is illustrated in Fig. 1. The real orientation of agent i is controlled by agent i's estimated orientation. The estimated orientation is updated by the local measurement (11). Since the real orientation affects the position of agents, if the real orientation converges to the prescribed desired orientation, the group of agents is going to achieve the desired formation rotated by the desired orientation. It shows that the formation shape converges to the desired formation of SE(2) via global orientation control.

3 Distributed formation control of SE(2) via global orientation control

3.1 Global orientation estimation

The dynamics of orientation for each agent can be written as follows:

$$\dot{R}_i = \Lambda_i R_i \tag{12}$$

where $\Lambda_i(t) \in \mathbb{R}^{2\times 2}$ is a skew-symmetric matrix for angular velocity of ith agent.

Let $\hat{R}_i(t) \in \mathbb{R}^{2\times 2}$ be agent i's estimated orientation. We design the dynamics of estimated orientation as follows:

$$\dot{\hat{R}}_i(t) = \sum_{j \in \mathscr{N}_i} \frac{(R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t))}{\| (R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha}} + \Lambda_i(t)\hat{R}_i(t) \tag{13}$$

where α is a positive value in $(0, 1)$. Note that the $\hat{R}_j$ is available to the agent i under the communication graph $\mathscr{G}_d^c$. For the analysis of the stability, we consider a coordinate transformation as $\hat{Q}_i = R_i^{\mathrm{T}}\hat{R}_i$. The derivative of $\hat{Q}_i$ is written as follows :

$$\begin{aligned} \dot{\hat{Q}}_i(t) &= \dot{R}_i^{\mathrm{T}}\hat{R}_i + R_i^{\mathrm{T}}\dot{\hat{R}}_i \\ &= -R_i^{\mathrm{T}}\Lambda_i\hat{R}_i + R_i^{\mathrm{T}}\dot{\hat{R}}_i \\ &= \sum_{j \in \mathscr{N}_i} \frac{R_j^{\mathrm{T}}\hat{R}_j(t) - R_i^{\mathrm{T}}\hat{R}_i(t)}{\| R_{ij}\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha}}. \end{aligned} \tag{14}$$

Since $\| R_{ij}\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha} = \| R_i^{\mathrm{T}}\hat{R}_i(t) - R_j^{\mathrm{T}}\hat{R}_j(t)) \|_F^{\alpha}$, (14) can be rewritten as follows:

$$\dot{\hat{Q}}_i(t) = \sum_{j \in \mathscr{N}_i} \frac{\hat{Q}_j(t) - \hat{Q}_i(t)}{\| \hat{Q}_i(t) - \hat{Q}_j(t)) \|_F^{\alpha}}. \tag{15}$$

Let $\hat{Q}$ be a stacked matrix form defined by $\hat{Q} := (\hat{Q}_1, \ldots, \hat{Q}_n)$. Equation (15) can be written in terms of $\hat{Q}$ as follows: the matrix form can be written as follows:

$$\dot{\hat{Q}} = -(L_c \otimes I_2)\hat{Q} \tag{16}$$

where the matrix $L_c = [l_{ij}^c] \in \mathbb{R}^{n\times n}$ can be defined as

$$l_{ij}^c = \begin{cases} 0, \ \text{if } (i, j) \in \mathscr{E}^c,\ i \neq j,\ \hat{Q}_i = \hat{Q}_j \ \text{or}\ (i, j) \notin \mathscr{E}^c,\ i \neq j \\ -1/\ \| \hat{Q}_i - \hat{Q}_j \|_F^{\alpha}, \ \text{if } (i, j) \in \mathscr{E}^c,\ i \neq j,\ \hat{Q}_i \neq \hat{Q}_j \\ \sum_{k \in \mathscr{N}_i} l_{ik}^c, \ \text{if } i = j. \end{cases}$$

From (16), we obtain the following theorem under Assumption 1.

Theorem 1: Under Assumption 1, $\hat{Q}(t)$ globally converges to $1_n \otimes \hat{Q}_1(0)$ in finite time.

Proof: See Section 8.1 of Appendix. □

From Theorem 1, we note that a finite value of $\hat{Q}$ is determined by an initial value of $\hat{Q}_1$. To ensure the convergence of estimated orientation in SO(2), the initial value of $\hat{R}_i$ should be chosen in the subspace of SO(2). Then, we can obtain the following corollary from the fact.

Corollary 1: Under orientation estimation law (13) and Assumption 1, there exists $R_e \in \mathrm{SO}(2)$ such that $\hat{R}_i, \forall i \in \mathscr{V}$ converges to R_iR_e in finite time if and only if $\hat{R}_1(0) \in \mathrm{SO}(2)$ is non-singular.

Proof: In Theorem 1, $\hat{Q}_i(t) \in \mathbb{R}^{2\times 2}$ converges to the initial value of $\hat{Q}_1 \in \mathbb{R}^{2\times 2}$ regardless of Λ_i, or i.e. $\hat{Q}_i(t) \to \hat{Q}_1(0)\ \forall i \in \mathscr{V}$ as

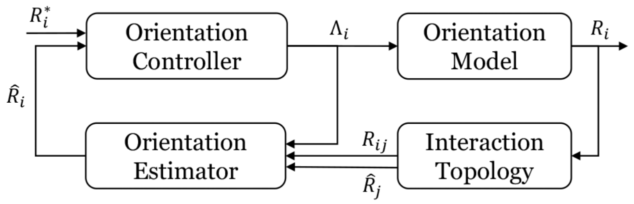

Fig. 2 *Block diagram for orientation estimation and control*

$t \to T_o$. Since $\hat{Q}_1(0) = R_1^{\mathrm{T}}(0)\hat{R}_1(0)$, if there is $\hat{R}_1(0) \in \mathrm{SO}(2)$ then $\hat{R}_i(t) \to R_i(t)\hat{Q}_1(0) \in \mathrm{SO}(2)\, \forall i \in \mathscr{V}$ as $t \to T_o$. □

3.2 Distributed orientation control

The orientation control law of (12) for each agent is proposed as follows:

$$\Lambda_i = \frac{1}{2}((\bar{R}_i)^{\mathrm{T}} - \bar{R}_i), \tag{17}$$

where $\bar{R}_i = \hat{R}_i(R_i^*)^{\mathrm{T}} \in \mathbb{R}^{2\times 2}$. Let us consider the equilibrium of the orientation system (12) with the control law (17). From the result of Corollary 1, we see that there exists a common rotation matrix R_e for all agents such that $\hat{R}_i$ converges to R_iR_e in finite time. The steady-state behaviour of the system (13) is, then, considered by substituting $\hat{R}_i$ with R_iR_e. It follows that the equilibrium of the system (12) satisfies

$$(R_i(t)R_e(R_i^*)^{\mathrm{T}} - R_i^*(R_e)^{\mathrm{T}}(R_i(t))^{\mathrm{T}})R_i(t) = 0. \tag{18}$$

There are two solutions of R_i for the equilibrium of the system (12) such as $R_i = (R_e)^{\mathrm{T}}R_i^*$ and $R_i = -(R_e)^{\mathrm{T}}R_i^*$. We show the stability analysis of equilibrium in the following result.

Lemma 1: For the system (12) with the orientation control law (17), the equilibrium point $R_i = -(R_e)^{\mathrm{T}}R_i^*$ is unstable.

Proof: See Section 8.2 of Appendix. □

Remark 1: Although the estimated orientation $\hat{R}_i$ is not in $SO(2)$ for all times under the estimation law (13), the orientation control law (17) is still in the tangent space of $SO(2)$.

The following theorem states that the equilibrium point $R_i = (R_e)^{\mathrm{T}}R_i^*$ for the dynamics (12) is exponentially stable.

Theorem 2: Under the dynamics (12) with the proposed estimation law (13) and control law (17), there is a common rotation matrix R_e such that R_i converges to $(R_e)^{\mathrm{T}}R_i^*$, $\forall i \in \mathscr{V}$, exponentially.

Proof: Define the error $E_i = R_e(R_i^*)^{\mathrm{T}}R_i$, then, the convergence $R_i \to (R_e)^{\mathrm{T}}R_i^*$ is equivalent to the convergence of $E_i \to I_2$. The derivative of E_i is $\dot{E}_i = R_e(R_i^*)^{\mathrm{T}}\dot{R}_i = R_e(R_i^*)^{\mathrm{T}}\Lambda_iR_i$, for all $i \in \mathscr{V}$. Since R_e, $R_i^* \in \mathrm{SO}(2)$, the derivative of E_i can be rewritten as $\dot{E}_i = \Lambda_iE_i$. Thus, the angle dynamics of E_i is written as follows:

$$\dot{\theta}_i^e = d\psi_{R_i}\{\Lambda_iE_i\} = -\sin(\bar{\theta}_i) \tag{19}$$

where θ_i^e and $\bar{\theta}_i$ are angles corresponding to E_i and $\bar{R}_i$, respectively. Since the angle dynamics of E_i is equivalent to the angle dynamics of R_i, we obtain that the error dynamics has an unstable equilibrium point in $\bar{\theta}_i = \pi$, which is proved in Lemma 1.

Consider the Lyapunov function $V_i = \frac{1}{2}\parallel E_i - I_2 \parallel_F^2 = \frac{1}{2}\mathrm{tr}((E_i - I_2)^{\mathrm{T}}(E_i - I_2))$ which is positive definite and continuously differentiable. Moreover, $V_i = 0$ if and only if $E_i = I_2$. The derivative of V_i is given by

$$\begin{aligned} \dot{V}_i &= \mathrm{tr}((E_i - I_2)^{\mathrm{T}}(\dot{E}_i)) \\ &= \mathrm{tr}(R_i^{\mathrm{T}}\Lambda_iR_i - R_e(R_i^*)^{\mathrm{T}}\Lambda_iR_i) \\ &= -\frac{1}{2}\mathrm{tr}(R_e(R_i^*)^{\mathrm{T}}(\bar{R}_i^{\mathrm{T}} - \bar{R}_i)R_i) \\ &= -\frac{1}{2}\mathrm{tr}(R_e(R_i^*)^{\mathrm{T}}\bar{R}_i^{\mathrm{T}}R_i - R_e(R_i^*)^{\mathrm{T}}\bar{R}_iR_i)\,. \end{aligned} \tag{20}$$

From Corollary 1, we obtain that the estimated orientation globally converges to its orientation in finite time T_o, i.e. $\bar{R}_i = \hat{R}_i(R_i^*)^{\mathrm{T}} \to R_iR_e(R_i^*)^{\mathrm{T}}$. Then, the derivative of V_i can be rewritten as follows:

$$\begin{aligned} \dot{V}_i &= -\frac{1}{2}\mathrm{tr}(I_2 - E_i^2) \\ &= -2(1 - \cos(\theta_i^e))(1 + \cos(\theta_i^e)) \le 0, \quad \forall t > T_o\,. \end{aligned} \tag{21}$$

Since $V_i = 2(1 - \cos\theta_i^e)$, (21) is rewritten as follows:

$$\dot{V}_i = -\underbrace{\left(2 - \frac{1}{2}V_i\right)}_{:=\bar{V}_i}V_i\,. \tag{22}$$

Since $\bar{V}_i \ge 0$ and there is an unstable equilibrium point when $\bar{V}_i = 0$, we can obtain a positive constant ε as follows:

$$\inf_{t \in [T_o, \infty)} \bar{V}_i := \varepsilon > 0. \tag{23}$$

Consequently, the derivative of V_i is rewritten as

$$\dot{V}_i \le -\varepsilon V_i, \quad \forall t > T_o\,. \tag{24}$$

Equation (24) shows that the convergence of the error E_i to zero is exponentially fast. □

From the result of Theorem 2, we see that $R_i(R_j)^{\mathrm{T}}$ converges to $R_i^*(R_j^*)^{\mathrm{T}}$ in exponential rate.

The block diagram for orientation estimation and control is illustrated in Fig. 2. Orientation controller is designed based on the estimated orientation.

3.3 Distributed formation control via orientation control

Consider the position dynamics of ith agent under the assumption of Problem 2. The following formation control law for ith agent is proposed:

$$\dot{p}_i^i = \sum_{j \in \mathscr{N}_i}((p_j^i - p_i^i) - R_i^*(p_j^* - p_i^*)), \quad i \in \mathscr{V} \triangleq \{1, \ldots, n\} \tag{25}$$

where p_i^i is the position of ith agent with respect to ${}^i\Sigma$. Since $\dot{p}_i^i = R_i\dot{p}_i$ and $R_i = E_i(R_e)^{\mathrm{T}}R_i^*$, we can rewrite the position dynamics in the view of global reference frame as follows:

$$\begin{aligned} \dot{p}_i &= \sum_{j \in \mathscr{N}_i}((p_j - p_i) - (R_i)^{\mathrm{T}}R_i^*(p_j^* - p_i^*)) \\ &= \sum_{j \in \mathscr{N}_i}((p_j - p_i) - (E_i)^{\mathrm{T}}R_e(p_j^* - p_i^*)) \\ &= \sum_{j \in \mathscr{N}_i}((p_j - p_i) - R_e(p_j^* - p_i^*)) \\ &\quad + \sum_{j \in \mathscr{N}_i}(R_e(I_2 - (E_i)^{\mathrm{T}})(p_j^* - p_i^*)) \end{aligned} \tag{26}$$

where E_i exponentially converges to I_2 by orientation control.

Let us define w_i as

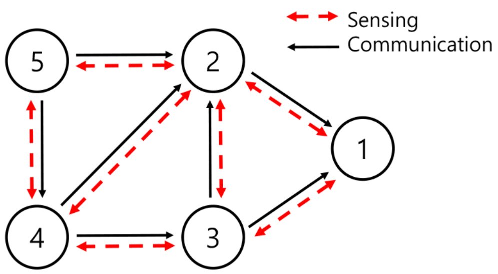

Fig. 3 *Interaction topology and the desired formation shape*

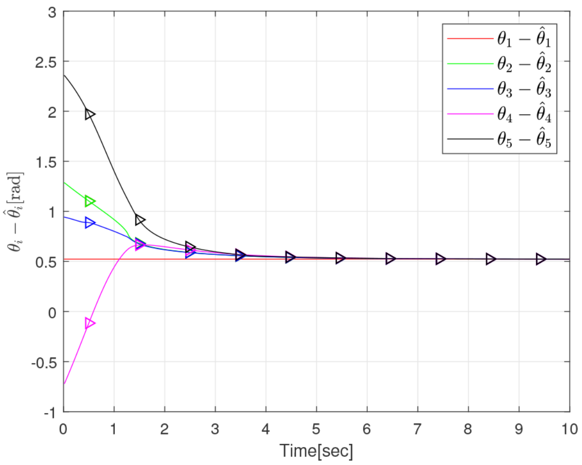

Fig. 4 *Error of global orientations in orientation estimation. The angle θ_i and $\hat{\theta}_i$ correspond to R_i and $\hat{R}_i$, respectively*

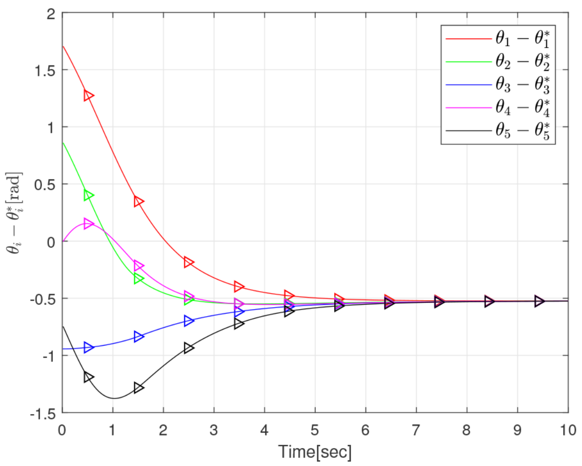

Fig. 5 *Error of global orientations in orientation control. The angle θ_i^* corresponds to R_i^**

$$w_i \triangleq \sum_{j \in \mathcal{N}_i} (R_e(I_2 - (E_i)^{\mathrm{T}})(p_j^* - p_i^*)) . \tag{27}$$

Using w_i and the fact that $R_e \in \mathrm{SO}(2)$, the norm of w_i can be written as

$$\begin{aligned} \| w_i \| &\le \sum_{j \in \mathcal{N}_i} \| R_e \| \| I_2 - (E_i)^{\mathrm{T}} \| \| p_j^* - p_i^* \| \\ &\le \sum_{j \in \mathcal{N}_i} \| I_2 - (E_i)^{\mathrm{T}} \| \| p_j^* - p_i^* \| . \end{aligned} \tag{28}$$

Since $\| I_2 - (E_i)^{\mathrm{T}} \|$ converges to zero exponentially, we can obtain that

$$\| w_i \| \le k_c e^{-\lambda_c(t - t_0)} \| p_j^* - p_i^* \| \tag{29}$$

where $k_c, \lambda_c > 0$. Then, (26) can be rewritten as follows:

$$\dot{p}_i = \sum_{j \in \mathcal{N}_i} ((p_j - R_e p_j^*) - (p_i - R_e p_i^*)) . \tag{30}$$

Let $\mathbf{C}_i \triangleq p_i - R_e p_i^*$ and $\mathbf{C} = [(\mathbf{C}_1)^{\mathrm{T}}, (\mathbf{C}_2)^{\mathrm{T}}, \ldots, (\mathbf{C}_n)^{\mathrm{T}}]^{\mathrm{T}}$. Equation (30) can be written as

$$\dot{\mathbf{C}} = -(L_k \otimes I_2)\mathbf{C} \tag{31}$$

where L_k is the Laplacian matrix under the assumption of Problem 2. Equation (31) implies that agent i's position converges to a finite point as $t \to \infty$

$$\lim_{t \to \infty} p_i(t) = R_e p_i^* + \mathbf{C}^*, i \in \mathscr{V} \triangleq \{1, \ldots, n\} \tag{32}$$

where $\mathbf{C}^*$ is a common value by the consensus algorithm. The solution of $\mathbf{C}(t)$ is written as

$$\mathbf{C} = \Phi(t, t_0)\mathbf{C} + \int_{t_0}^{t} \Phi(t, \tau)W(\tau)\mathrm{d}\tau \tag{33}$$

where $W(\tau) = [w_1^{\mathrm{T}}, w_2^{\mathrm{T}}, \ldots, w_n^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{\Phi}$ is the state transition matrix. Let us define a equilibrium set $\Xi \triangleq \{\xi \in \mathbb{R}^{2n} | \xi_1 = \xi_2 = \cdots = \xi_n\}$. Then, we can see that $\mathbf{C}$ converges to the equilibrium set in an exponential rate if $W = 0$. Consequently, we obtain the following theorem which is proved in [9].

Theorem 3: Under the formation control law (25) via orientation control, the formation $\mathbf{p}^i(t) = (p_1^i(t), \ldots, p_n^i(t))$ exponentially converges to the desired formation rotated by the desired orientation such that $\mathbf{p}^i$ converges to $R_i^* \mathbf{p}^*$ exponentially.

Using (32) and Theorem 3, Problem 2 results in

$$\begin{aligned} p_{ij}^i &= p_i^i - p_j^i \\ &= R_i^*(R_e)^{\mathrm{T}}(R_e p_i^* + \mathbf{C}^*) - R_i^*(R_e)^{\mathrm{T}}(R_e p_j^* + \mathbf{C}^*) \\ &= R_i^*(p_i^* - p_j^*) \end{aligned} \tag{34}$$

which means that the formation shape is allowed to a rotation and a translation comparing with the desired formation. It shows agents characterised in $\mathrm{SO}(2) \times \mathbb{R}^2 (= \mathrm{SE}(2))$ are controlled by proposed control laws (17) and (25). Moreover, the agents only use local measurement p_{ij}^i to control both the orientation and the formation.

4 Simulation

To verify the proposed strategies, we provide the simulation results. For the simulation, five agents modelled by the single-integrator system have the interaction topology illustrated in Fig. 3. The initial orientation angles and positions of agents are assigned randomly. The desired orientation and formation are, respectively, given as $\theta^* = \{\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*, \theta_5^*\} = \{1.18, 0.57, -2.51, -1.34, 2\}$ and $\mathbf{p}^* = \{(1.5, 2.86)^{\mathrm{T}}, (1, 2)^{\mathrm{T}}, (2, 2)^{\mathrm{T}}, (2, 1)^{\mathrm{T}}, (1, 1)^{\mathrm{T}}\}$. The shape of desired formation is illustrated in Fig. 3.

Fig. 4 illustrates that errors of estimated orientation converge to a common value 0.52[rad]. It implies that the estimated orientation converges to actual orientation with a common offset R_e. Fig. 5 illustrates the result of orientation control. The simulation shows that the proposed orientation control method achieves convergence of orientation to a desired orientation with the common offset, i.e. $\{\theta_1, \ldots, \theta_5\} \to \{0.66, 0.05, -3.03, -1.86, 1.48\} = \theta^* - 0.52$[rad]. It implies that the relative orientation of real orientations converges to its desired relative orientation in exponential rate.

Fig. 6 illustrates a simulation of the proposed formation control strategy. In this figure, we set the initial formation as $\mathbf{p}(0) = \{(3, 3)^{\mathrm{T}}, (1, 2)^{\mathrm{T}}, (2, 1)^{\mathrm{T}}, (5, 2)^{\mathrm{T}}, (1, 1)^{\mathrm{T}}\}$. The formation of agents converges to $\{(3, 3)^{\mathrm{T}}, (2.74, 3.97)^{\mathrm{T}}, (2.03, 3.26)^{\mathrm{T}}, (1.33, 3.97)^{\mathrm{T}}, (2.03, 4.67)^{\mathrm{T}}\}$. The simulation shows that agents form the desired formation shape illustrated in Fig. 3,

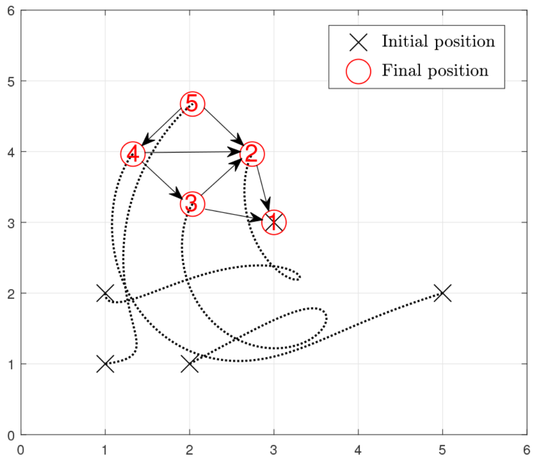

Fig. 6 *Displacement-based formation control of SE(2)*

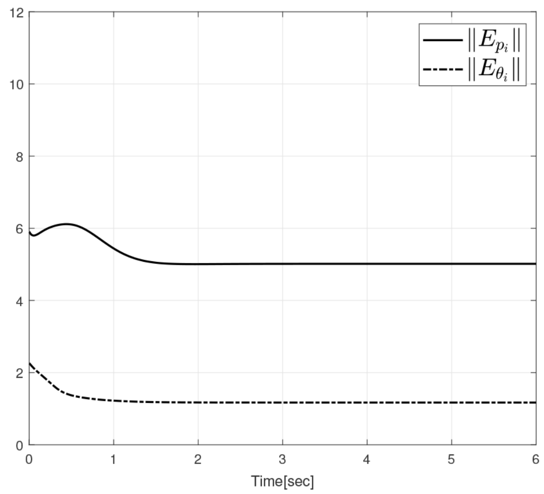

Fig. 7 *Error of the formation and the orientation of agents*

although agents are rotated to the desired relative orientation. Since the formation shape is achieved up to a translation and the rotation compared with the desired formation $\mathbf{p}^*$, it accounts for the formation control of the two-dimensional special Euclidean group SE(2) via orientation control. The formation error $\| E_{p_i} \|$ and the orientation error $\| E_{\theta_i} \|$ are illustrated in Fig. 7. Since both the formation and the orientation of agents converge to the prescribed desired value, $\| E_{p_i} \|$ and $\| E_{\theta_i} \|$ converge to a common state.

5 Conclusion

In this paper, we propose a distributed formation control strategy to achieve the formation shape with freedom of the two-dimensional special Euclidean group SE(2)(=SO(2)$\times \mathbb{R}^2$). The main feature of the proposed strategy is to consider the rotation of local reference frames by controlling agents' orientations. The real orientation of agents is controlled by estimated orientations with regard to ${}_g\Sigma$, where we use the distributed estimation strategy using only local measurement. The proposed strategy ensures the exponential convergence to the desired formation shape of SE(2). It implies that the fully distributed formation control of the special Euclidean group SE(2) is successfully achieved by orientation control. For future work, we are interested in applying the proposed strategy for multi-sensor network applications having surveillance and observation missions. Also, it is of interest to extend the proposed strategy to three-dimensional space.

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8 Appendix

8.1 Proof of Theorem 1

Theorem 1 can be proved from estimation results of [14]. Under Assumption 1, a root node (Agent 1) has no neighbour. Thus, the root node's estimated state maintains its initial state, i.e. $\hat{Q}_1(t) = \hat{Q}_1(0) = R_1^{\mathrm{T}}(0)\hat{R}_1(0)$. The convergence of the other estimated states is proved as follows.

First, since Agent 2 has only one neighbour, Agent 1, the estimated state can be written as follows:

$$\dot{\hat{Q}}_2(t) = \frac{\hat{Q}_1(t) - \hat{Q}_2(t)}{\| \hat{Q}_2(t) - \hat{Q}_1(t)) \|_F^{\alpha}}. \tag{35}$$

Let us consider a Lyapunov function as $V_2(t) = (1/2) \| \hat{Q}_2(t) - \hat{Q}_1(t)) \|_F^2$. The derivative is written as follows:

$$\begin{aligned}\dot{V}_2(t) &= \mathrm{tr}((\hat{Q}_2(t) - \hat{Q}_1(t))^{\mathrm{T}}(\dot{\hat{Q}}_2(t) - \dot{\hat{Q}}_1(t))) \\ &= -\gamma_2 V_2(t)^{(2-\alpha)/2}\end{aligned} \tag{36}$$

where $\gamma_2 = (2/2^{\alpha/2})$. It shows that $\hat{Q}_2(t) \to \hat{Q}_1(0)$ as $t \to T_2$ where $T_2 \le 2V_2(0)^{\alpha/2}/(\gamma_2\alpha)$.

Next, we suppose that $\hat{Q}_i = \hat{Q}_1(0)$ for $t \ge T_i$, $\forall i \in \{1, \dots, n-1\}$. Let us consider Agent k having one or some neighbours in $\{1, \dots, k-1\}$ where $\forall k \in \{3, \dots, n\}$. Then, $\hat{Q}_k = \hat{Q}_1(0)$ for $t \ge T_k$ where $T_k \ge T_i' \triangleq \max_{i \in \mathcal{N}_k}(T_i)$. The estimation dynamics (15) can be rewritten as follows:

$$\dot{\hat{Q}}_k(t) = -\sum_{i \in \mathcal{N}_k} \frac{\hat{Q}_k(t) - \hat{Q}_i(t)}{\| \hat{Q}_k(t) - \hat{Q}_i(t)) \|_F^{\alpha}}. \tag{37}$$

Consider a Lyapunov function as $V_k(t) = (1/2)\sum_{i \in \mathcal{N}_k} \| \hat{Q}_k(t) - \hat{Q}_i(t)) \|_F^2$. The derivative of V_k is written as follows:

$$\dot{V}_k(t) = \sum_{i \in \mathcal{N}_k} \mathrm{tr}((\hat{Q}_k(t) - \hat{Q}_i(t))^{\mathrm{T}}(\dot{\hat{Q}}_k(t) - \dot{\hat{Q}}_i(t))). \tag{38}$$

Since $V_k(t)$ is bounded in finite time, the states of V_k are bounded for $t \le T_i'$. For $t > T_i'$, $\dot{\hat{Q}}_i = 0$ and $\hat{Q}_i = \hat{Q}_1$, $\forall i \in \mathcal{N}_k$. Then, the derivative of V_k can be rewritten as follows:

$$\begin{aligned}\dot{V}_k(t) &= |\mathcal{N}_k| \sum_{i \in \mathcal{N}_k} \mathrm{tr}((\hat{Q}_k(t) - \hat{Q}_i(t))^{\mathrm{T}}\dot{\hat{Q}}_k(t)) \\ &= -\gamma_k V_k(t)^{(2-\alpha)/2}\end{aligned} \tag{39}$$

where $\gamma_k = 2|\mathcal{N}_k|/2^{\alpha/2}$. It shows that $\hat{Q}_k(T_k) = \hat{Q}_1(0)$ for $t \ge T_k$ where $T_k \le T_i' + 2V_k(T_i')^{\alpha/2}/(\gamma_k\alpha)$.

Finally, we implement the process until $k = n$. It shows that $\hat{Q}(t)$ globally converges to $1_n \otimes \hat{Q}_1(0)$ in finite time.

8.2 Proof of Lemma 1

From the work of [27], we can prove Lemma 1 as follows. Let us define the tangent mapping $d\psi_x\{g(x)\} \triangleq \frac{g(x)^{\mathrm{T}}}{\| x \|} R_{\frac{\pi}{2}} \frac{x}{\| x \|}$, where

$$R_{\frac{\pi}{2}} = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix}.$$

By using the tangent mapping, the system dynamics of R_i and $\hat{R}_i$ are rewritten as follows:

$$\begin{aligned}\dot{\theta}_i &= d\psi_{R_i}\{\Lambda_i R_i\} \\ \dot{\hat{\theta}}_i &= d\psi_{\hat{R}_i}\{\Lambda_i \hat{R}_i\} + d\psi_{\hat{R}_i}\left\{\sum_{j \in \mathcal{N}_i} \frac{(R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t))}{\| (R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha}}\right\} \\ &= -\sin(\bar{\theta}_i) + \sum_{j \in \mathcal{N}_i} \frac{(R_{ij}\hat{R}_j - \hat{R}_i)^{\mathrm{T}}}{\| (R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha} \| \hat{R}_i \|} R_{\frac{\pi}{2}} \frac{\hat{R}_i}{\| \hat{R}_i \|} \\ &= -\sin(\bar{\theta}_i) + \sum_{j \in \mathcal{N}_i} \left(\frac{(R_{ij}\hat{R}_j - \hat{R}_i)^{\mathrm{T}}}{\| (R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha}} R_{\frac{\pi}{2}} \frac{\hat{R}_i}{\| \hat{R}_i \|^2} \right)\end{aligned}$$

where $\| (R_{ij}(t)\hat{R}_j(t) - \hat{R}_i(t)) \|_F^{\alpha}$ is a positive scalar. Since $\hat{R}_i^{\mathrm{T}} R_{\frac{\pi}{2}} \hat{R}_i$ is identically zero and $a^{\mathrm{T}}b = \| a \| \| b \| \cos(\angle a - \angle b)$, where the operator $\angle$ is an angle of a vector, this system dynamics can be simplified as

$$\begin{aligned}\dot{\theta}_i &= -\sin(\bar{\theta}_i) \\ \dot{\hat{\theta}}_i &= -\sin(\bar{\theta}_i) + \sum_{j \in \mathcal{N}_i} c_{ij}\cos\left(\theta_{ij} + \hat{\theta}_j - \hat{\theta}_i - \frac{\pi}{2}\right)\end{aligned} \tag{40}$$

$$= -\sin(\bar{\theta}_i) + \sum_{j \in \mathcal{N}_i} c_{ij}\sin(\theta_{ij} + \hat{\theta}_j - \hat{\theta}_i) \tag{41}$$

where c_{ij} is a positive value.

To examine the stability behaviour of the equilibrium point corresponding to Lemma 1, suppose that $\bar{\theta}_i = \pi$, $\hat{\theta}_i = \pi + \theta_i^* + \theta^e$, and $\theta_i = \pi + \theta_i^* - \theta^e$, where θ_i^* and θ^e are the angle of R_i^* and R_e, respectively. The Jacobian matrix of the system dynamics is derived as follows:

$$\begin{aligned}J &= \begin{bmatrix} -\cos(\bar{\theta}_i) - a_{ij} & a_{ij} \\ -\cos(\bar{\theta}_i) & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\cos(\bar{\theta}_i) - a_{ij} & a_{ij} \\ -\cos(\bar{\theta}_i) & 0 \end{bmatrix}\Bigg|_{\bar{\theta}_i = \pi, \hat{\theta}_i = \pi + \theta_i^* + \theta^e, \theta_i = \pi + \theta_i^* - \theta^e} \\ &= \begin{bmatrix} 1 - \sum_{j \in \mathcal{N}_i} c_{ij} & \sum_{j \in \mathcal{N}_i} c_{ij} \\ 1 & 0 \end{bmatrix}\end{aligned} \tag{42}$$

where $a_{ij} = \sum_{j \in \mathcal{N}_i} c_{ij}\cos(\theta_{ij} + \hat{\theta}_j - \hat{\theta}_i)$. The eigenvalues of the Jacobian matrix (42) are easy to check as 1 and $-\sum_{j \in \mathcal{N}_i} c_{ij}$. It follows that the equilibrium point corresponding to Lemma 1 is unstable.