



Quantum entanglement in inflationary cosmology

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Abstract We investigate the time-dependent entanglement entropy in the AdS space with a dS boundary which represents an expanding spacetime. On this time-dependent spacetime, we show that the Ryu–Takayanagi formula, which is usually valid in the static spacetime, provides a leading contribution to the time-dependent entanglement entropy. We also study the leading behavior of the entanglement entropy between the visible and invisible universes in an inflationary cosmology. The result shows that the quantum entanglement monotonically decreases with time and finally saturates a constant value inversely proportional to the square of the Hubble constant. Intriguingly, we find that even in the expanding universes, the time-dependent entanglement entropy still satisfies the area law determined by the physical distance.

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1 Introduction

Recently, considerable attention has been paid to the quantum entanglement which is one of the important physical concepts to figure out important quantum features of a variety of physical system. Although the entanglement entropy is conceptually well defined in the quantum field theory (QFT) [1–4], calculating it for interacting QFTs is usually a difficult and formidable task. In this situation, holography recently proposed in the string theory [5–8] allows us to evaluate such a nontrivial entanglement entropy nonperturbatively even in strongly interacting systems [9–18]. By applying the holographic technique, in this work, we investigate the time-dependent quantum entanglement of an expanding system and inflationary cosmology.

In order to calculate the entanglement entropy, one first divides a system into two subsystems, A and B , and then evaluate the reduced density matrix of A defined as the trace over B . In this case, two subsystems are divided by an entangling surface and an observer living in A cannot receive any information from B . This situation is very similar to the black hole [19,20]. An observer living at the asymptotic boundary cannot get any information from the inside of the black hole. Because of this similarity, there were many attempts to understand the Bekenstein–Hawking entropy in terms of the entanglement entropy [21–33]. Furthermore, the similarity between the black hole horizon and the entangling surface has led to a new and fascinating holographic formula to calculate the entanglement entropy on the dual gravity side. Although the holographic method has not been proved yet, it was checked that the holographic formula perfectly reproduces the known results of a two-dimensional conformal field theory (CFT) [34–40].

In a cosmological model described by a dS space [41,42], there exists a specific surface called the cosmic event horizon. An observer living at the center of a dS space cannot see the outside of the cosmic event horizon and, moreover,

the cosmic event horizon radiates similar to the black hole horizon. From the quantum entanglement point of view, the cosmic event horizon naturally divides a universe into two parts. One is a visible universe which we can see in future and the other is called an invisible universe. In this case, the invisible universe means that an observer living in a visible universe cannot see the outside of the cosmic event horizon even after infinite time evolution. Although the visible and invisible universes are casually disconnected from each other, the quantum correlation between them can still exist. Therefore, it would be interesting to investigate the quantum entanglement between the visible and invisible universes, which may give us new insights about the outside of our visible universe and the effect of the invisible universe on the cosmology of the visible universe.

In order to investigate the quantum entanglement between two subsystems in the expanding universe, we take into account an AdS space with a dS boundary space [43,44] which has also been studied by others from the different viewpoint [45,46]. The minimal surface extended to such an AdS space corresponds to the entanglement entropy of an expanding space defined at the boundary of the AdS space [41]. Since we take into account a time-dependent geometry, we need to consider the covariant formulation [11] instead of the Ryu–Takayanagi (RT) formula defined on a constant time slice [9,10]. However, we show that the RT formula with a fixed time gives rise to the leading contribution to the covariant formulation. Even in this case, the time dependence of the subsystem size leads to a nontrivial time dependence of the entanglement entropy. Consequently, the RT formula with the time-dependent entangling region can well approximate the leading term of the covariant entanglement entropy. In this work, we check this points analytically and numerically by comparing the results of the RT and covariant formulas.

Before studying the entanglement entropy of a visible universe, we first consider a subsystem whose boundary expands in time, unlike the cosmic event horizon. In this case, the entanglement entropy in the early time era increases by the square of the cosmological time τ , whereas it in the late time era grows up exponentially by $e^{(d-2)H\tau}$ for a d -dimensional QFT. If we take the cosmic event horizon as an entangling surface, the entanglement entropy shows a totally different behavior. The cosmic event horizon at $\tau = 0$ is located at the equator of a $(d - 1)$ -dimensional sphere and monotonically decreases as the cosmological time elapses. In the late inflation era, the cosmic event horizon approaches a constant value proportional to the inverse of Hubble constant. Similarly, the corresponding entanglement entropy also monotonically decreases and approaches a constant value. In the present work, intriguingly, we find that the time-dependent entanglement entropy in the expanding universe still satisfies the area law determined by not the comoving distance but the physical distance.

The rest of this paper is organized as follows: In Sect. 2, we briefly review an AdS space with a dS boundary. On this background, we study the entanglement entropy of an expanding system for $d = 2, 3, 4$ cases in Sect. 3. In Sect. 4, we introduce a cosmic event horizon and divide a universe into visible and invisible universes and then, study the quantum correlation between the visible and invisible universes in the inflationary cosmology. Finally, we finish this work with some concluding remarks in Sect. 5.

2 AdS space with a dS boundary

Consider a $(d + 1)$ -dimensional AdS space which can be embedded into a $(d + 2)$ -dimensional flat manifold with two time signatures. Denoting the $(d + 2)$ -dimensional flat metric as

$$ds^2 = -dY_{-1}^2 - dY_0^2 + \delta_{ij}dY^i dY^j, \quad (2.1)$$

where i and j run from 1 to d , the Lorentz group of this $(d + 2)$ -dimensional flat space is given by $SO(2, d)$. In order to obtain a $(d + 1)$ -dimensional AdS metric, we impose the following constraint

$$-R^2 = -Y_{-1}^2 - Y_0^2 + \delta_{ij}Y^i Y^j. \quad (2.2)$$

Then, the hypersurface satisfying the constraint represents a $(d + 1)$ -dimensional AdS space with an AdS radius R . Since the imposed constraint is also invariant under the $SO(2, d)$ transformation, the resulting AdS geometry becomes a $(d + 1)$ -dimensional space invariant under the $SO(2, d)$ transformation which is nothing but the isometry group of the AdS space. There exist a variety of parametrization satisfying the above constraint. In this work, we focus on the parametrization which allows a d -dimensional de Sitter (dS) space at the boundary. Now, let us parametrize the coordinates of the ambient space as [41]

$$Y_{-1} = R \cosh \frac{\rho}{R}, \quad Y_0 = R \sinh \frac{\rho}{R} \sinh \frac{t}{R} \quad \text{and} \\ Y^i = R n^i \sinh \frac{\rho}{R} \cosh \frac{t}{R}, \quad (2.3)$$

where n^i indicates a d -dimensional orthonormal vector satisfying $\delta_{ij}n^i n^j = 1$. The resulting AdS metric then gives rise to

$$ds^2 = d\rho^2 + \sinh^2 \left(\frac{\rho}{R} \right) \\ \left[-dt^2 + R^2 \cosh^2 \left(\frac{t}{R} \right) \left(d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2 \right) \right], \quad (2.4)$$

where $d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2$ indicates a metric of a $(d - 1)$ -dimensional unit sphere. According to the AdS/CFT correspondence, the boundary of this AdS space defined at $\rho = \infty$ can be regarded as the space-time we live in. Above the

boundary metric shows a dS space which can describes an inflationary cosmology.

In order to divide the boundary space into two subsystems, let us first assume that we are at $\theta = 0$ and that two subsystems are bordered at θ_o . For convenience, we call the subsystem we are in is an observable system and the other subsystem an unobservable system. In general, the border is called the entangling surface in the entanglement entropy study. Although we do not get any information from the unobservable system, the quantum state of the observable system can be affected from the unobservable system due to the non-trivial quantum entanglement. Let us assume that the entire system is described by a pure state $|\Psi\rangle$ represented by the product of two subsystem's states [9, 10, 18, 34–36, 47–49]

$$|\Psi\rangle = |\psi\rangle_o |\psi\rangle_u, \tag{2.5}$$

where $|\psi\rangle_o$ and $|\psi\rangle_u$ indicate the states of the observable and unobservable systems, respectively. Then, the reduced density matrix of the observable system is given by tracing over the unobservable part

$$\rho_o = \text{Tr}_u |\Psi\rangle \langle\Psi|, \tag{2.6}$$

and the entanglement entropy is defined by the Von Neumann entropy

$$S_E = -\text{Tr}_o \rho_o \log \rho_o. \tag{2.7}$$

Although the entanglement entropy is conceptually well defined in a quantum field theory, it is not easy to calculate it in general cases. Even for those cases, we can apply the covariant or RT formula and get more information about the entanglement entropy. In this work, following the holographic proposition, we will discuss the entanglement entropy of the expanding system in Eq. (2.4).

3 Entanglement entropy on the expanding system

Above, we have discussed the entanglement entropy between the observable and unobservable systems. However, it is not still clear how we can divide the observable and unobservable systems. One simple choice is to take a constant θ_o . Under this simple choice, the size of the two subsystems gradually increases as time elapses. The set-up with a constant θ_o may be useful to describe an expanding material or to figure out the entanglement entropy of a time-dependent subsystem. On the other hand, it is also interesting to take into account the time-dependent entangling surface. In this section, we first investigate the entanglement entropy defined by a constant θ_o and then discuss further the entanglement entropy of inflationary cosmology in the next section. As will be discussed in the next section, eternal inflation usually has a horizon outside of which we can not measure classically. The existence of such a horizon can seriously affect the entanglement

entropy in the late time era. In this section, anyway, we focus on the entanglement entropy of an expanding material like an expanding quark-gluon plasma in the RHIC experiments. The expanding material system is usually too small to compare the horizon of our universe so that we do not need to consider the effect of the horizon in this section.

3.1 Comments on the entanglement entropy in the expanding system

Before investigating the entanglement entropy, we need to discuss several important issues appearing in the time-dependent geometry. For simplicity, let us first focus on the $d = 2$ case which can give us a simple solvable toy model. For $d = 2$, the dual geometry reduces to the three-dimensional AdS space

$$ds^2 = d\rho^2 + \sinh^2\left(\frac{\rho}{R}\right) \left[-dt^2 + R^2 \cosh^2\left(\frac{t}{R}\right) d\theta^2\right]. \tag{3.1}$$

If we look at the boundary space of this AdS space with a fixed ρ , the boundary metric has the form of the cosmological-type metric depending on time. In order to describe the entanglement entropy on the time-dependent background, we assume that the observable system is in the range of

$$-\frac{\theta_o}{2} \leq \theta \leq \frac{\theta_o}{2}. \tag{3.2}$$

For convenience, we can also introduce a new coordinate $R/z = \sinh(\rho/R)$. In terms of the new coordinate, the previous AdS₃ metric can be reexpressed as

$$ds^2 = \frac{R^2 dz^2}{z^2(1+z^2/R^2)} + \frac{R^2}{z^2} [-dt^2 + R^2 \cosh^2(t/R) d\theta^2]. \tag{3.3}$$

Here, we assume that the AdS boundary is located at $z = \epsilon \ll 1$. Then, the value of θ_o specifies the size of the subsystem. More precisely, the subsystem size l is given by

$$l(t) = \frac{R^2}{\epsilon} \cosh(t/R) \theta_o. \tag{3.4}$$

Due to the time-dependent background geometry, the minimal surface corresponding to the entanglement entropy is described by the following covariant formula [11]

$$S_E = \frac{R}{4G} \int_{-\frac{\theta_o}{2}}^{\frac{\theta_o}{2}} d\theta \frac{1}{z} \sqrt{\frac{z'^2}{1+z^2/R^2} - t'^2 + R^2 \cosh^2(t/R)}, \tag{3.5}$$

where the prime indicates a derivative with respect to θ .

The smoothness of the minimal surface and the invariance under $\theta \rightarrow -\theta$ requires $z' = 0$ and $t' = 0$ at the turning point z_* and, consequently, the minimal surface extends only to $0 \leq z \leq z_*$. Since the above action does not explicitly depend on θ , there exists a conserved charge

$$H = -\frac{R^2 \cosh^2(t/R)}{4Gz\sqrt{\frac{z'^2}{1+z^2/R^2} - t'^2 + R^2 \cosh^2(t/R)}}. \tag{3.6}$$

At the turning point, the conserved charge simply reduces to

$$H = -\frac{R \cosh(t_*/R)}{4Gz_*}, \tag{3.7}$$

where t_* denotes the value of t at the turning point. Comparing these two relations, we obtain

$$z' = \pm \frac{\sqrt{R^2 + z^2} \sqrt{z^2 [t'^2 - R^2 \cosh^2(t/R)] + R^2 z_*^2 \cosh^4(t/R) \operatorname{sech}^2(t_*/R)}}{Rz}. \tag{3.8}$$

This relation shows that, when z approaches 0 at the boundary, z' must diverge regardless of the value of t' . In the holographic entanglement entropy calculation, as will be seen, the main contribution usually comes from a small z region, so that it is important to know the asymptotic behavior of t' to determine the leading behavior of the entanglement entropy. For example, if the asymptote of t' is proportional to z' , we must take into account the contribution of t' at leading order. On the other hand, if the value of t' at the boundary does not diverge, the contributions of t' can be regarded as a higher order correction and the leading contribution comes only from z' . As a consequence, knowing the asymptotic behavior of t' becomes important to determine the leading behavior of the entanglement entropy.

In order to see the asymptote of t' , we rewrite t' as a function of z' by combining the above two relations

$$t' = \pm \frac{R\sqrt{z^2 z'^2 - (R^2 + z^2) \cosh^2(t/R) \left(\frac{z_*^2 \cosh^2(t/R) \operatorname{sech}^2(t_*/R) - z^2}{z\sqrt{R^2 + z^2}} \right)}}{z\sqrt{R^2 + z^2}}. \tag{3.9}$$

This relation does not allow us to fix the asymptotic form of t' because of opposite signs of two leading terms even for $z \rightarrow 0$. In order to determine the asymptotic form of t' , we need to investigate further the equation of motion for z . When replacing t' and t'' with z, z' and z'' by using Eq. (3.9), the equation of motion for z is generally given by a very complicated form. In order to determine the asymptotic form of t' , however, it is sufficient to see the leading behavior of the obtained complicated equation. To do so, let us first consider the scaling behavior of z at the boundary which is useful to pick up the equation governing the leading contribution. When z approaches 0 at the boundary, z' must diverges as $1/z$. This fact implies that z should involve the factor like $\delta^{1/2} = \sqrt{\theta_o - \theta}$. At the boundary, therefore, z and its relatives must be scaled by $z \rightarrow \delta^{1/2} z, z' \rightarrow \delta^{-1/2} z'$,

and $z'' \rightarrow \delta^{-3/2} z''$ at leading order. Note that z' and z'' can also have other terms differently scaled by $z' \rightarrow \delta^{-a} z'$ with $a < 1/2$ and $z'' \rightarrow \delta^{-b} z''$ with $b < 3/2$. However, those terms generally lead to higher order corrections. Substituting the above leading scaling behaviors into the equation of motion, we finally obtain the leading equation governing the asymptotic behavior of z

$$0 = z^3 z'' + R^2 z_*^2 \operatorname{sech}^2(z_*/R) \cosh^4(t_o/R), \tag{3.10}$$

where t_o indicates time at the boundary, $t(\theta_o)$. This equation allows the following general solution

$$z = \frac{\sqrt{z_*^2 \cosh^4(t_o/R) \operatorname{sech}^2(t_*/R) - c_1^4 R^2 (c_2 + \theta)^2}}{c_1}, \tag{3.11}$$

where c_1 and c_2 are two integral constants. The constraint, $z = 0$ at $\theta = \theta_o$, determines one of parameters in terms of the others

$$t_* = R \cosh^{-1} \left(\frac{z_* \cosh^2(t_o/R)}{R\sqrt{c_1^4 \theta_o^2 + 2c_2 c_1^4 \theta_o + c_2^2 c_1^4}} \right). \tag{3.12}$$

Substituting t_* and the solution z in Eq. (3.11) into Eq. (3.9) and taking the limit $z \rightarrow 0$, we finally see that t' has the following value at the boundary

$$t' = R\sqrt{\cosh^2\left(\frac{t_o}{R}\right) - c_1^4 (c_2 + \theta_o)^2 - c_1^2}. \tag{3.13}$$

This result shows that t' is finite at the boundary. Therefore, t'^2 in Eq. (3.5) leads to only higher order corrections for a three-dimensional AdS space.

Above we showed that t'^2 in the covariant formula (3.5) does not contribute the leading behavior of the time-dependent entanglement entropy. Therefore, if we are interested only in the leading behavior of the entanglement entropy, we can set $t'^2 = 0$ in Eq. (3.5). This fact implies that the leading entanglement entropy can be calculated by taking a constant t in the covariant formulation. Since the covariant formulation at a fixed t is reduced to the RT formula, the above discussion indicates that the RT formula can be a good approximation of the covariant formulation at least at leading order. Even in this case, the leading entanglement entropy still has a nontrivial time-dependence because the entangling region is generally time-dependent in the expanding universe.

3.2 Time-dependence of the leading entanglement entropy

In the previous section, we showed that the RT formula can be a good approximation of the covariant formulation. In this section, we will analytically calculate the time-dependent

entanglement entropy by using the RT formula and then compare the result of the RT formula with the numerical result of the covariant formula. We will also show that these two results are well matched, as mentioned before. After this consistency check, in the next sections, we will study the time-dependent entanglement entropy of a four-dimensional inflationary universe.

Let us first consider the case with a constant θ_o . Then, $\pm\theta_o/2$ correspond to two boundaries of the observable system. Since we took a constant θ_o , the size of the observable system usually expands in the expanding system. More precisely, the size of the observable system is given by $\sinh(\Lambda/R) \cosh(t/R) R\theta_o$ at the AdS boundary denoted by $\rho = \Lambda$. This shows that the size of the observable system increases by $\cosh(t/R)$. Since $\cosh(t/R)$ is invariant under the time reversal, from now on we take into account only the non-negative time period, $0 \leq t < \infty$. This implies that the observable system begins the expansion at $t = 0$. If we take Λ as an infinity, it usually leads to a divergence which is related to a UV divergence of the dual field theory. In this case, Λ is usually introduced to regularize the UV divergence. From now on, we consider Λ as a large but finite value. This finite Λ can be related to the renormalized energy scale which may be associated with the energy scale of the expanding universe.

In order to get more physical intuition about the entanglement entropy on the time-dependent geometry, let us consider several particular limits. We first define the turning point as ρ_* which corresponds to the minimum value extended by the minimal surface in the ρ -coordinate. In the case with $\rho_*/R \gg 1$, we can calculate the entanglement entropy analytically but perturbatively even for higher dimensional cases. This parameter range corresponds to the UV limit and may give rise to a good guide line to figure out physical implication of the numerical study. Ignoring higher order corrections caused by the t'^2 term as explained before, the leading contribution to the entanglement entropy for $\rho_*/R \gg 1$ is governed by [50–54]

$$S_E = \frac{1}{4G} \int_{-\theta_o/2}^{\theta_o/2} d\theta \sqrt{\rho'^2 + \frac{R^2}{4} e^{2\rho/R} \cosh^2(t/R)}. \quad (3.14)$$

Solving the equation of motion derived from it, θ_o at a given t is determined by the turning point

$$\theta_o = \frac{4}{e^{\rho_*/R} \cosh(t/R)}. \quad (3.15)$$

When θ_o and t are given, the turning point is inversely determined as a function of θ_o and t

$$e^{\rho_*/R} = \frac{4}{\theta_o} \frac{1}{\cosh(t/R)}. \quad (3.16)$$

Note that t/R must not be large in order to obtain a large ρ_*/R . This fact implies that the approximation with $\rho_*/R \gg$

1 is valid only in the early time era. In addition, this result shows that the turning point moves into the interior of the AdS space as time evolves.

Performing the integral of the entanglement entropy with the obtained solution, the resulting entanglement entropy reduces to

$$S_E = \frac{[(\Lambda - \rho_*) + R \log 2]}{2G}. \quad (3.17)$$

This result together with Eq. (3.16) shows that the entanglement entropy increases by t^2 for $t/R \ll 1$

$$S_E \sim \frac{R \log \theta_o + \Lambda - R \log 2}{2G} + \frac{t^2}{4GR}. \quad (3.18)$$

If $t/R > 1$, on the other hand, it increases linearly in time

$$S_E \sim \frac{R \log \theta_o + \Lambda - 2R \log 2}{2G} + \frac{t}{2G}. \quad (3.19)$$

Now, let us take into account a more general case without the constraint $\rho_*/R \gg 1$. The general form of the entanglement entropy reads from Eq. (3.1)

$$S_E = \frac{1}{4G} \int_{-\theta_o/2}^{\theta_o/2} d\theta \sqrt{\rho'^2 + R^2 \sinh^2(\rho/R) \cosh^2(t/R)}. \quad (3.20)$$

After solving the equation of motion, performing the integral gives rise to

$$\begin{aligned} \frac{\theta_o}{2} &= \int_{\rho_*}^{\infty} d\rho \frac{\sinh(\frac{\rho_*}{R})}{R \cosh(\frac{t}{R}) \sinh(\frac{\rho}{R}) \sqrt{\sinh^2(\frac{\rho}{R}) - \sinh^2(\frac{\rho_*}{R})}} \quad (3.21) \\ &= \frac{1}{\cosh(t/R)} \left[\frac{\pi}{2} - \arctan(\sinh(\rho_*/R)) \right]. \end{aligned}$$

Rewriting it leads to the following relation

$$\sinh(\rho_*/R) = \cot\left(\frac{\theta_o \cosh(t/R)}{2}\right), \quad (3.22)$$

which reproduces the previous result in Eq. (3.16) for $\rho_*/R \gg 1$. In the general case, the resulting entanglement entropy reads

$$S_E = \frac{\Lambda}{2G} + \frac{R \log \left[\sin\left(\frac{1}{2}\theta_o \cosh(t/R)\right) \right]}{2G}. \quad (3.23)$$

When $\theta_o \cosh(t/R) \ll 1$, this result again reproduces the previous results obtained in the early inflation era. Notice that the bulk geometry has the shift symmetry, $\theta \rightarrow \theta + \Delta\theta$. If we choose the subsystem size as θ_o , this symmetry causes to change the size of the subsystem. This fact certainly shows that the entanglement entropy depends on the subsystem size. For $d = 2$, the boundary dS spacetime in the global patch is homeomorphic to $R \times S^1$ where R indicates the time direction and the spatial part is given by a circle which is periodic. For this reason, the periodicity naturally appears in the entanglement entropy.

It is worth noting that the resulting entanglement entropy is well defined only in the time range of $0 \leq t < t_f$, where t_f satisfies $\theta_o \cosh(t_f/R) = 2\pi$. After this critical time t_f ,

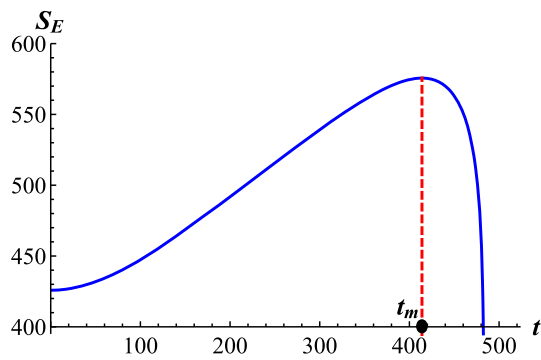


Fig. 1 We take $\epsilon = 1/1000$, $R = 100$, $\theta_o = 1/10$ and $G = 1$

the logarithmic term of the entanglement entropy is not well defined. Now, let us define another critical time t_m satisfying $\theta_o \cosh(t_m/R) = \pi$. At this critical time ($t = t_m$), the observable and unobservable systems have the same size. In this case, the turning point is located at $\rho_* = 0$ and the entanglement entropy has a maximum value, $S_E = \Lambda/(2G)$. Near t_m ($t < t_m$), the entanglement entropy approaches this maximum value slowly by $-(t_m - t)^2$

$$S_E \approx \frac{\Lambda}{2G} - \frac{\theta_o^2 \sinh^2(t_m/R)}{16GR} (t_m - t)^2 + \mathcal{O}((t_m - t)^4).$$

From these results, we can see that the entanglement entropy of the observable system increases by t^2 in the early time and saturates the maximum value at a finite time t_m . After t_m , the entanglement entropy rapidly decreases as shown in figure 1. Therefore, we can summarize the entanglement entropy of a two-dimensional expanding system as follows

- In the early time with $\rho_*/R \gg 1$ and $t/R \ll 1$, the entanglement entropy increases by t^2 .
- In the intermediate era with $\rho_*/R \gg 1$ and $t/R > 1$, the entanglement entropy increases linearly as time evolves.
- In the late time with $\rho_*/R \sim 0$ and $t \approx t_m$, the entanglement entropy slowly increases by $-(t_m - t)^2$ and finally saturates the maximum value at $t = t_m$.
- After t_m , the entanglement entropy rapidly decreases and vanishes after finite time evolution. Since the entanglement entropy must be positive, this fact implies that the quantum entanglement between two subsystems disappears after a finite time elapse.

In Fig. 1, we plot the exact entanglement entropy given in Eq. (3.23), which shows the expected time dependence of the previous analytic calculation.

It has been well known that the entanglement entropy of a two-dimensional CFT dual to an AdS_3 has a logarithmic divergence and its coefficient is proportional to the central charge of the dual CFT [9, 10]. However, the above result for AdS_3 with the dS_2 boundary shows a linear divergence

($\sim \Lambda$) instead of the logarithmic one. This is because the coordinate used in this work is different from the one usually used in Ref. [9, 10, 40]. This fact becomes manifest in the z -coordinate. In the UV limit ($z \rightarrow 0$) with $t/R \ll 1$, z is related to ρ by $e^{\rho/R} \sim R/z$ and the metric in Eq. (3.3) is reduced to

$$ds^2 \approx \frac{R^2 dz^2}{z^2} + \frac{R^2}{z^2} \left[-dt^2 + \frac{R^2}{4} d\theta^2 \right], \quad (3.24)$$

which is locally equivalent to the AdS space in the Poincare patch. Thus, the linear divergence appearing in Eq. (3.23) can be reinterpreted as a logarithmic one in the z -coordinate system, $\Lambda/R = -\log(\epsilon/R)$, where ϵ indicates the UV cut-off of the z -coordinate. As a result, the linear divergence obtained here is consistent with the known logarithmic one up to the coordinate transformation.

Above we studied the leading behavior of the time-dependent entanglement entropy by applying the RT formula. To check the validity of the RT formula as the leading approximation of the covariant formulation, we also calculated the entanglement entropy of the covariant formula numerically. Figure 2 shows that the analytic result of the RT formula is well matched to the numerical result of the covariant formulation, as mentioned before. This fact is an additional evidence for our previous prescription that the RT formula can give rise to the leading contribution to the time-dependent entanglement entropy.

3.3 Higher dimensional expanding systems

Now, let us consider the higher dimensional case with $d \geq 3$. In terms of the z coordinate, the previous $(d+1)$ -dimensional AdS metric in Eq. (2.4) can be rewritten as

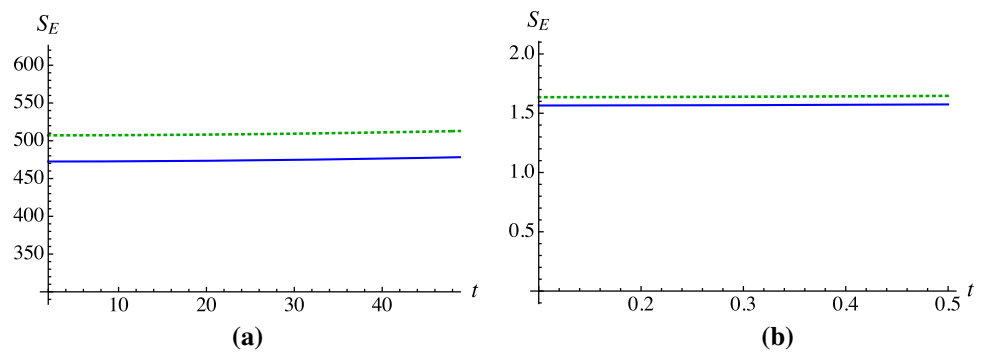
$$ds^2 = \frac{R^2 dz^2}{z^2(1+z^2/R^2)} + \frac{R^2}{z^2} \left[-dt^2 + R^2 \cosh^2(t/R) \left(d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2 \right) \right], \quad (3.25)$$

where the boundary is located at $z = 0$. On this background, the covariant holographic entanglement entropy is governed by

$$S_E = \frac{\Omega_{d-2} R^{2d-3} \cosh^{d-2}(t/R)}{4G} \int_0^{\theta_o} d\theta \frac{\sin^{d-2} \theta}{z^{d-1}} \sqrt{\frac{z^2}{1+z^2/R^2} - t^2 + R^2 \cosh^2(t/R)}, \quad (3.26)$$

where we take the range of θ as $0 \leq \theta \leq \theta_o$ instead of $-\theta_o/2 \leq \theta \leq \theta_o/2$. Varying this action, the configuration of a minimal surface is usually determined by two highly nontrivial differential equations. Note that in the higher dimensional case there is no well-defined conserved quantity because the above entanglement entropy explicitly depends on θ . Due to this reason, unfortunately, it is not easy to fix the value of t' at the boundary. Inevitably, a numerical study

Fig. 2 The blue curve represents an analytical result and the dotted green line represents a numerical one. The horizontal axis denotes the boundary time. Numerical input data: **a** $R = 100, \epsilon = 0.0008, \theta = 0.1993$ and $G = 1$. **b** $R = 1, \epsilon = 0.00005, \theta = 1.5658$ and $G = 1$



is required to understand the time-dependent entanglement entropy in a higher dimensional expanding system.

Now, we ask whether we can still use the RT formula as a leading approximation of the above covariant formulation. Although we cannot determine the value of t' at the boundary, we can still argue that the RT formula is useful to understand the leading entanglement entropy even in the higher dimensional expanding system. Near the boundary with $z \rightarrow 0$, since t is finite, the leading behavior of the entanglement entropy is determined by the values of z' and t' . In this case, we can think of two situations. If t'/z' approaches zero at the boundary, the effect of t' can be ignored and the RT formula becomes a good leading approximation similar to the previous AdS₃ case. However, when $t'/z' = a$ at the boundary, we cannot ignore the effect of t' . Even in this case, a must be smaller than 1 to have a real entanglement entropy. This fact implies that the square root inside of the above entanglement entropy is again proportional to z' up to a constant multiplication, $\sqrt{1 - a^2}$. As a consequence, the resulting covariant entanglement entropy must be proportional to the result of the RT formula at leading order. In other words, the time dependence of the entanglement entropy at leading order can still be well described by the RT formula with $t' = 0$ up to a constant multiplication even in the higher dimensional case. Following this prescription, from now on we utilize the RT formula to determine the leading time dependence of the entanglement entropy in the expanding system.

We first discuss the entanglement entropy of the observable system in the early time with $t/R \ll 1$. Assuming that the observable system is very tiny in the early time, then we can take $\theta_o \ll 1$. In this case, the minimal surface is extended only to the UV region represented as $0 \leq z \leq z_*$ with $z_*/R \ll 1$. This is because z_*/R is usually proportional to θ_o at $t = 0$, as will be seen. Due to the small size of the observable system, the AdS metric in the early time can be well approximated by

$$ds^2 \approx \frac{R^2 dz^2}{z^2(1 + z^2/R^2)} + \frac{R^2}{z^2} [-dt^2 + R^2 \cosh^2(t/R) (d\theta^2 + \theta^2 d\Omega_{d-2}^2)]. \tag{3.27}$$

On this background, the entanglement entropy reduces to

$$S_E = \frac{\Omega_{d-2} R^{2d-3} \cosh^{d-2}(t/R)}{4G} \int_0^{\theta_o} d\theta \frac{\theta^{d-2}}{z^{d-1}} \sqrt{\frac{z'^2}{1 + z^2/R^2} + R^2 \cosh^2(t/R)}. \tag{3.28}$$

In order to find a perturbative solution satisfying $z/R \leq z_*/R \ll 1$, we introduce a small parameter λ for indicating the smallness of the solution. Then, the perturbative expansion of the solution can be parametrized as

$$z(\theta) = \lambda \left(z_0(\theta) + \lambda z_1(\theta) + \lambda^2 z_2(\theta) + \dots \right). \tag{3.29}$$

When varying this perturbative solution with respect to θ , it is worth noting that the derivative of the solution, $z'(\theta)$, must be expanded as

$$z'(\theta) = z'_0(\theta) + \lambda z'_1(\theta) + \lambda^2 z'_2(\theta) + \dots \tag{3.30}$$

This is because θ has the same order of z/R in the early time. Before performing the explicit calculation, let us think about the parity transformation, $z \rightarrow -z$ and $\theta \rightarrow -\theta$. Under this parity transformation, we can easily see that the metric in Eq. (3.27) and the entanglement entropy are invariant. If we transforms $\lambda \rightarrow -\lambda$ instead of z_n in Eq. (3.29), only z_{2n} terms give rise to the consistent transformation with $z \rightarrow -z$. Due to this reason, the z_{2n+1} terms automatically vanish. As a consequence, we can set $z_1(\theta) = 0$ without loss of generality.

At leading order of λ , the entanglement entropy is given by

$$S_0 = \frac{\Omega_{d-2} R^{2d-3} \cosh^{d-2}(t/R)}{4G} \int_0^{\theta_o} d\theta \frac{\theta^{d-2}}{z_0^{d-1}} \sqrt{z_0'^2 + R^2 \cosh^2(t/R)}. \tag{3.31}$$

In a higher dimensional theory unlike the $d = 2$ case, the entanglement entropy relies on θ explicitly. Thus, there is no well-defined conserved quantity, as mentioned before. This fact implies that we must solve the second order differential equation to obtain the entanglement entropy. At leading order, the minimal surface configuration can be determined

by solving the equation of motion derived from S_0

$$\begin{aligned} 0 = & \frac{2 \cosh^2(t/R) \theta z_0 z_0''}{R^2} + \frac{2(d-2) z_0 z_0'^3}{R^4} \\ & + \frac{2(d-1) \cosh^2(t/R) \theta z_0'^2}{R^2} \\ & + \frac{2(d-2) \cosh^2(t/R) z_0 z_0'}{R^2} \\ & + 2(d-1) \cosh^4(t/R) \theta. \end{aligned} \quad (3.32)$$

Despite the complexity of the equation of motion, it allows the following simple and exact solution regardless of the dimension d

$$\frac{z_0}{R} = \cosh(t/R) \sqrt{\theta_o^2 - \theta^2}. \quad (3.33)$$

From this, we see that the turning point denoted by z_* is proportional to θ_o , as mentioned before,

$$\frac{z_*}{R} = \theta_o \cosh(t/R). \quad (3.34)$$

Note that this relation is derived from the leading order of the entanglement entropy. If we further consider higher order corrections, the turning point can vary with some small corrections.

When a UV cut-off denoted by ϵ is given, we can easily see from the background metric that the volume of the observable system is given by

$$\mathcal{V}_{d-1} = \frac{\Omega_{d-2} R^{2(d-1)} \cosh^{d-1}(t/R)}{d-1} \frac{\theta_o^{d-1}}{\epsilon^{d-1}}, \quad (3.35)$$

while the area of the entangling surface becomes

$$\mathcal{A}_{d-2}(t) = \Omega_{d-2} R^{2(d-2)} \cosh^{d-2}(t/R) \frac{\theta_o^{d-2}}{\epsilon^{d-2}}. \quad (3.36)$$

These formulas show that the area of the entangling surface increases by $\cosh^{d-2}(t/R)$ as time evolves. At $t = 0$, in particular, the area reduces to

$$\bar{\mathcal{A}}_{d-2} = \Omega_{d-2} R^{2(d-2)} \frac{\theta_o^{d-2}}{\epsilon^{d-2}}, \quad (3.37)$$

which is determined by two parameters, ϵ and θ_o . In the holographic study, the minimal surface is extended only to $\epsilon \leq z \leq z_*$, so that $z_* > \epsilon$ must be satisfied for consistency. Recalling further that $z_*/R = \theta_o$ at $t = 0$, we finally obtain $\theta_o > \epsilon/R$. This fact implies that, when the expansion began at $t = 0$, the observable system and the entangling surface have the non-vanishing volume and area. In (3.36) and (3.37) the parameters θ_o and ϵ describe the size of the horizon and the energy scale of inflation universe, respectively. For this reason, it is an interesting topic whether it can compare the values of these parameters with the observational cosmological data. To do that, we need to extend the present work to a more realistic inflation model. Unfortunately, it is beyond

the scope of this paper. In this work, we focus on a qualitative feature of the entanglement entropy in the evolving universe.

Now, let us consider the $d = 3$ case. Using the perturbative expansion discussed before, the entanglement entropy is expanded into

$$S_E = S_0 + S_2 + \dots, \quad (3.38)$$

with

$$S_0 = \frac{\Omega_1 R^3 \cosh(t/R)}{4G} \int_0^{\theta_o - \theta_c} d\theta \frac{\theta}{z_0} \sqrt{z_0^2 + R^2 \cosh^2(t/R)}, \quad (3.39)$$

$$S_2 = -\frac{\Omega_1 R^4 \cosh(t/R)}{8G} \int_0^{\theta_o} d\theta \frac{\theta \left[z_0^3 z_0' + 4R^2 z_2 z_0'^2 - 2R^2 z_0 z_0' z_2' + 4R^4 z_2 \cosh^2(t/R) \right]}{R^3 z_0^3 \sqrt{z_0^2 + R^2 \cosh^2(t/R)}}, \quad (3.40)$$

where we set $\lambda = 1$ after the perturbative expansion and introduce θ_c as a UV cut-off in the θ -direction. In the second integral, θ_c was removed because it does not give any additional UV divergence. Substituting the leading order solution in Eq. (3.33) into S_0 and performing the integral, we finally obtain the leading contribution to the entanglement entropy

$$S_0 = \frac{\Omega_1 R^2 \sqrt{\theta_o}}{4\sqrt{2}G\sqrt{\theta_c}} - \frac{\Omega_1 R^2}{4G}. \quad (3.41)$$

The first correction caused by $z_2(\theta)$ is determined by the following differential equation

$$\begin{aligned} 0 = & z_2'' + \frac{(\theta_o^2 - 2\theta^2)}{\theta(\theta_o^2 - \theta^2)} z_2' - \frac{2\theta_o^2}{(\theta_o^2 - \theta^2)^2} z_2 \\ & + 2R\sqrt{\theta_o^2 - \theta^2} \cosh^3(t/R). \end{aligned} \quad (3.42)$$

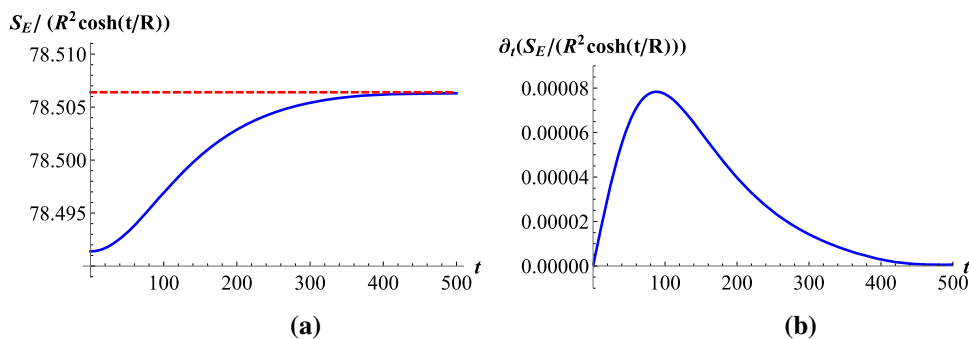
This equation allows an exact solution

$$\begin{aligned} z_2 = & c_2 - \frac{c_2 \theta_o \tanh^{-1} \left(\frac{\sqrt{\theta_o^2 - \theta^2}}{\theta_o} \right)}{\sqrt{\theta_o^2 - \theta^2}} \\ & + \frac{6c_1 + (\theta^4 - 4\theta_o^2 \theta^2 + 3\theta_o^4 + 4\theta_o^4 \log \theta) R \cosh^3(t/R)}{6\sqrt{\theta_o^2 - \theta^2}}, \end{aligned} \quad (3.43)$$

where c_1 and c_2 are two integral constants. These two integral constants must be fixed by imposing two appropriate boundary conditions. The natural boundary conditions are $z_2(\theta_o) = 0$ and $z_2'(0) = 0$. The first condition implies that the end of the minimal surface is located at the boundary, while the second constraint is required to obtain a smooth minimal surface at $\theta = 0$. These two boundary conditions determine two integral constants to be

$$\begin{aligned} c_1 = & -\frac{2\theta_o^4 R \log \theta_o \cosh^3(t/R)}{3}, \\ c_2 = & -\frac{2\theta_o^3 R \cosh^3(t/R)}{3}. \end{aligned} \quad (3.44)$$

Fig. 3 We take $\epsilon = 1/1000$, $R = 100$, $\theta_o = 1/20$ and $G = 1$



Substituting the found perturbative solutions again into S_2 , the first correction to the entanglement entropy is given by

$$S_2 = -\frac{5\theta_o^2 \Omega_1 R^2 \cosh^2(t/R)}{36G}. \tag{3.45}$$

Above the regulator θ_c is usually associated with the regulator ϵ in the z -direction. Using the perturbative solution we found, θ_c can be represented as a function of ϵ

$$\theta_c = \frac{\epsilon^2}{2\theta_o R^2 \cosh^2(t/R)} - \frac{2\epsilon^3}{9R^3 \cosh(t/R)} + \mathcal{O}(\epsilon^4) \tag{3.46}$$

As a consequence, the resulting perturbative entanglement entropy leads to

$$S_E = \frac{\theta_o \Omega_1 R^3 \cosh(t/R)}{4G\epsilon} - \frac{\Omega_1 R^2 (\theta_o^2 \cosh^2(t/R) + 3)}{12G} + \mathcal{O}(\epsilon). \tag{3.47}$$

Recalling the formula in Eq. (3.37), this entanglement entropy can be rewritten as

$$S_E = \frac{\mathcal{A}_1(t)R}{4G} - \frac{\Omega_1 R^2 (\theta_o^2 \cosh^2(t/R) + 3)}{12G} + \mathcal{O}(\epsilon), \tag{3.48}$$

where $\mathcal{A}_1(t)$ indicates the area of the entangling surface at a given time t . The leading contribution to the entanglement entropy, as expected, satisfies the area law even in the time-dependent space. Expanding it further in the early time, the entanglement entropy leads to

$$S_E = \frac{\bar{\mathcal{A}}_1 R}{4G} - \frac{\Omega_1 R^2}{4G} - \frac{\theta_o^2 \Omega_1 R^2}{12G} + \left(\frac{\bar{\mathcal{A}}_1}{8GR} - \frac{\theta_o^2 \Omega_1}{12G} \right) t^2 + \mathcal{O}(t^4), \tag{3.49}$$

where $\bar{\mathcal{A}}_1 = \mathcal{A}_1(0)$. This result shows that the entanglement entropy in the early time increases by t^2

$$S_E(t) - S_E(0) \approx \left(\frac{\bar{\mathcal{A}}_1}{8GR} - \frac{\theta_o^2 \Omega_1}{12G} \right) t^2. \tag{3.50}$$

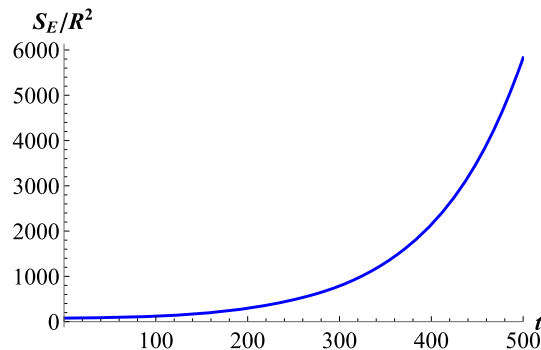


Fig. 4 We take $\epsilon = 1/1000$, $R = 100$, $\theta_o = 1/20$ and $G = 1$

It also shows that the increase of the entanglement entropy is proportional to the area of the entangling surface at leading order.

In order to see the entanglement entropy in the late time, we must go beyond the perturbative expansion. After finding a numerical solution satisfying Eq. (3.32), we investigate how the corresponding entanglement entropy increases in time. In Fig. 3, we depict the value of $S_E / (R^2 \cosh(t/R))$ and its time derivative. In Fig. 3a, the value of $S_E / (R^2 \cosh(t/R))$ approaches a constant in the late time. This fact becomes manifest in Fig. 3b, where the time derivative of $S_E / (R^2 \cosh(t/R))$ approaches zero in the late time. Consequently, we can see that the entanglement entropy increases exponentially ($S_E \sim e^{t/R}$) in the late time (see Fig. 4).

Repeating the same calculation for $d = 4$, the entanglement entropy of the $d = 4$ observable system, similar to the $d = 3$ case, increases by t^2 in the early time and exponentially grows in the late time. In the late time, the increase of the entanglement entropy is proportional to $S_E \sim e^{t/R}$ for $d = 3$ and $S_E \sim e^{2t/R}$ for $d = 4$ which becomes manifest in Fig. 5. These results imply that the entanglement entropy of the expanding observable system increases by t^2 in the early time regardless of d and in the late time grows by $S_E \sim e^{(d-2)t/R}$ for a general d . For the black hole formation corresponding to the thermalization of the dual field theory, the entanglement entropy usually increases by t^2 in the early time similar to the

expanding observable system. However, in the late time of the thermalization the entanglement entropy is saturated and becomes a thermal entropy, while the entanglement entropy of the expanding observable system increases exponentially in the late time.

4 Entanglement entropy of the visible universe in the inflationary cosmology

In the previous section, we studied the quantum entanglement of the expanding observable system which is described by the constant θ_o . In this section, we investigate the entanglement entropy of the visible universe in the inflationary cosmology. In an inflationary model, there exists a natural border dividing the entire universe into two parts. Because of the growing scale factor in the inflationary model, there exists an invisible universe which we cannot see forever. On the other hand, the universe we can see is called the visible universe and the boundary of the visible universes is called cosmic event horizon which corresponds to the border of the visible and invisible universes. In this case, the invisible universe is casually disconnected from us. Due to the existence of the natural border of two universes in the inflationary model, it would be interesting to study the quantum correlation between them. In this section, we will investigate such an entanglement entropy for a four-dimensional inflationary cosmology.

Let us first define cosmic event horizon as the boundary of the visible universe. From Eq. (3.25) for $d = 4$, the boundary metric reads at $z = \epsilon$

$$ds_B^2 = \frac{R^2}{\epsilon^2} [-dt^2 + R^2 \cosh^2(t/R) (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2)], \quad (4.1)$$

which describes $\mathbf{R}^+ \times \mathbf{S}^3$. In order to interpret the boundary metric as the cosmological one, we introduce a cosmological time τ and Hubble constant H such that

$$\tau = \frac{R}{\epsilon} t \quad \text{and} \quad H = \frac{\epsilon}{R^2}, \quad (4.2)$$

where the Hubble constant H has a slightly different value from the usual 4-dimensional dS cosmology. In holography, the overall warping factor of the AdS space contributes to determining the Hubble constant because of the boundary dS metric is reduced from one-dimensional higher AdS space. Then, the boundary metric reduces to the one representing an inflationary cosmology

$$ds_B^2 = -d\tau^2 + \frac{\cosh^2(H\tau)}{H^2} (d\theta^2 + \sin^2 \theta d\Omega_{d-2}^2), \quad (4.3)$$

where the scale factor is given by $a(\tau) = \cosh(H\tau)/H$. Due to the nontrivial scale factor, the distance travelled by light is restricted to a finite region whose boundary by definition corresponds to cosmic event horizon. More precisely, cosmic

event horizon in the above cosmological metric is determined by

$$d(\tau) = a(\tau) \int_t^\infty \frac{c d\tau'}{a(\tau')} \\ = \left[\frac{\pi}{2} - 2 \arctan \left(\tanh \frac{H\tau}{2} \right) \right] \frac{\cosh H\tau}{H}, \quad (4.4)$$

where the light speed was taken to be $c = 1$. In Fig. 6a, we plot how the cosmic event horizon changes as the cosmological time τ evolves. In the early inflation era, the cosmic event horizon decreases with time, whereas it approaches a constant value $1/H$ in the late inflation era which is a typical feature of the dS space.

The existence of cosmic event horizon indicates that the visible universe, the inside of cosmic event horizon, is casually disconnected from the invisible universe, the outside of cosmic event horizon [55–58]. In other words, if we are at the center of the visible universe, we can never receive any information from the invisible universe. Even in this situation, there can exist a nontrivial quantum correlation between them, which can be measured by the entanglement entropy. From the viewpoint of the entanglement entropy, cosmic event horizon naturally plays a role of an entangling surface which divides a system into two subsystems.

To go further, let us reexpress the cosmic event horizon in terms of the angle appearing in the AdS space. For distinguishing the cosmic event horizon from the previous expanding entangling surface parametrized by θ_o , we use a different symbol θ_v which is given by a function of τ unlike θ_o . Assuming that we are at the north pole of the three-dimensional sphere denoted by $\theta = 0$, our visible universe can be characterized by $0 \leq \theta \leq \theta_v$. In this case, the radius of the entangling surface is determined from the AdS metric

$$l = \int_0^{\theta_v} d\theta \frac{\cosh H\tau}{H} = \frac{\theta_v \cosh H\tau}{H}. \quad (4.5)$$

Because the radius of the entangling surface must be identified with cosmic event horizon, the comparison between them determines θ_v as a function of the cosmological time

$$\tan \left(\frac{\pi}{4} - \frac{\theta_v}{2} \right) = \tanh \frac{H\tau}{2}. \quad (4.6)$$

This result shows that θ_v start with $\pi/2$ at $\tau = 0$ and gradually decreases to 0 at $\tau = \infty$ with a fixed subsystem size l . In the late inflation era, the cosmic event horizon becomes a constant independent of the cosmological time, $d(\tau) = 1/H$. In Fig. 6b, we plot θ_v relying on the cosmological time. In the figure, θ_v starts from $\pi/2$ at $\tau = 0$ and rapidly decreases to 0 as the cosmological time goes on.

By using θ_v we found, it is possible to calculate holographically the entanglement entropy of the visible universe. Before performing the calculation, it is worth noting that the cosmological time and the Hubble constant are defined

Fig. 5 We take $\epsilon = 1/1000$, $R = 100$, $\theta_o = 1/20$ and $G = 1$

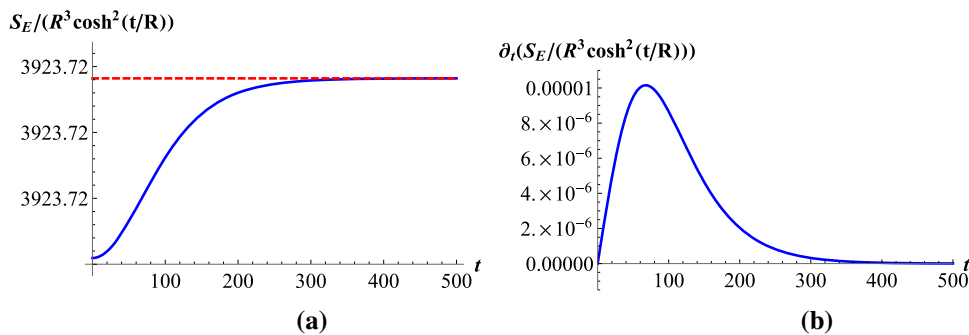
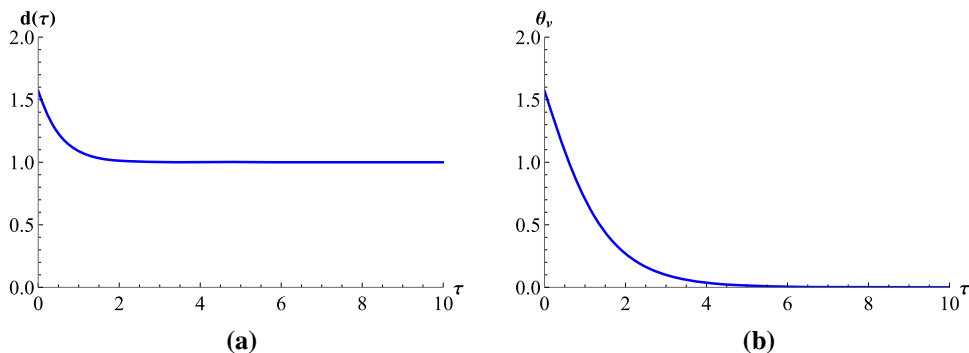


Fig. 6 cosmic event horizon relying on the cosmological time τ where we take $H = 1$



only at the boundary. The minimal surface corresponding to the entanglement entropy is extended to the bulk of the dual geometry, so that we cannot exploit the definition of τ and H in the course of calculating the area of the minimal surface. After the calculation, however, we can replace t and ϵ with τ and H through Eq. (4.2). This is because the resulting area of the minimal surface represents the entanglement entropy defined at the boundary at which τ and H are well defined.

4.1 Entanglement entropy at $\tau = 0$

For simplicity, let us first consider the entanglement entropy at $\tau = 0$. Using the relation in Eq. (4.2), $\tau = 0$ implies $t = 0$ regardless of ϵ . For $d = 4$, the holographic entanglement entropy formula is given by Eq. (3.26) with $t = 0$ and θ_v instead of θ_o . If we alternatively take into account θ as a function of z , the corresponding entanglement entropy in the inflationary model can be rewritten as

$$S_E = \frac{\Omega_2 R^5}{4G} \int_{\epsilon}^{\infty} dz \frac{\sin^2 \theta}{z^3} \sqrt{R^2 \dot{\theta}^2 + \frac{1}{1 + z^2/R^2}}, \quad (4.7)$$

where the dot indicates a derivative with respect to z . Deriving the equation of motion from this action, it allows a specific solution which satisfies $\dot{\theta} = 0$ and furthermore $\theta = \pm\pi/2$. This solution indicates an equatorial plane of S^3 . Performing the above integral with this equatorial plane solution, we finally obtain

$$S_E = \frac{\Omega_2 R^5}{4G} \left(\frac{1}{2\epsilon^2} - \frac{1}{2R^2} \log \frac{2R}{\epsilon} + \frac{1}{4R^2} \right). \quad (4.8)$$

If we interpret ϵ as the UV cut-off, this result shows the power-law divergence together with the logarithmic divergence, as expected in the entanglement entropy calculation for $d = 4$. Rewriting ϵ in terms of H by using Eq. (4.2), we finally obtain the following entanglement entropy at $\tau = 0$

$$S_E = \frac{\Omega_2 R}{8GH^2} - \frac{\Omega_2 R^3}{8G} \log \frac{2}{HR} + \frac{\Omega_2 R^3}{16G}. \quad (4.9)$$

4.2 Entanglement entropy in the late inflation era

In the inflationary cosmology unlike the previous expanding system, the perturbative calculation of the entanglement entropy is possible in the late inflation era because θ_v becomes small at large t or τ . In the late inflation era we can apply the previous perturbative expansion of z . Using the perturbation of z , the leading contribution and the first correction to the entanglement entropy are given by

$$S_0 = \frac{\Omega_2 R^5 \cosh^2(t/R)}{4G} \int_0^{\theta_v - \theta_c} d\theta \frac{\theta^2}{z_0^3} \sqrt{z_0'^2 + R^2 \cosh^2(t/R)}, \quad (4.10)$$

$$S_2 = -\frac{\Omega_2 R^6 \cosh^2(t/R)}{8G} \int_0^{\theta_v - \theta_c} d\theta \frac{\theta (z_0^3 z_0'^2 + 6R^2 z_2 z_0'^2 - 2R^2 z_0 z_0' z_2' + 6R^4 z_2 \cosh^2(t/R))}{z_0^4 R^3 \sqrt{z_0'^2 + R^2 \cosh^2(t/R)}}. \quad (4.11)$$

Note that unlike the $d = 3$ case, the upper limit of the integral range in S_2 has θ_c . This is because we need to reintroduce θ_c to regularize an additional divergence appearing in S_2 for $d = 4$.

Substituting the leading solution in Eq. (3.33) into S_0 , we obtain the following leading contribution to the entanglement entropy

$$S_0 = \frac{\theta_v R^3 \Omega_2}{16G\theta_c} - \frac{\Omega_2 R^3}{16G} \log \frac{2\theta_v}{\theta_c} - \frac{\Omega_2 R^3}{32G}. \quad (4.12)$$

In this result, we can see that, when $\theta_c \rightarrow 0$, the leading contribution leads to the expected power-law and logarithmic divergences for $d = 4$.

Now, let us consider the deformation of the minimal surface described by z_2 , which is governed by the following differential equation

$$0 = z_2'' + \frac{2}{\theta} z_2' - \frac{3\theta_v^2}{(\theta_v^2 - \theta^2)^2} z_2 + \frac{(3\theta_v^2 - 2\theta^2) R \cosh^3(t/R)}{\sqrt{\theta_v^2 - \theta^2}}. \quad (4.13)$$

This equation allows us to find the following exact solution

$$z_2 = \frac{c_1 (\theta_v - \theta)^2}{\theta \sqrt{\theta_v^2 - \theta^2}} + \frac{c_2}{\sqrt{\theta_v^2 - \theta^2}} + \frac{(\theta^5 - 5\theta^2\theta^3 - 2\theta_v^3\theta^2 - 2\theta^4\theta - 2\theta_v^5) R \cosh^3(t/R)}{6\theta \sqrt{\theta_v^2 - \theta^2}} + \frac{\{(\theta_v + \theta)^2 \log(\theta_v + \theta) - (\theta_v - \theta)^2 \log(\theta_v - \theta)\} \theta_v^3 R \cosh^3(t/R)}{2\theta \sqrt{\theta_v^2 - \theta^2}}, \quad (4.14)$$

where c_1 and c_2 are two integration constants. Imposing two boundary condition, $z_2(\theta_v) = 0$ and $z_2'(0) = 0$ discussed in the previous section, c_1 and c_2 are determined to be

$$c_1 = \frac{\theta_v^3 R \cosh^3(t/R)}{3}, \quad c_2 = \frac{\theta_v^4 R \cosh^3(t/R) [5 - 6 \log(2\theta_v)]}{3}. \quad (4.15)$$

Substituting the obtained solutions into S_2 again and performing the integral result in

$$S_2 = -\frac{3\theta_v^2 \Omega_2 R^3 \cosh^2(t/R)}{32G} \log \frac{2\theta_v}{\theta_c} + \frac{11\theta_v^2 \Omega_2 R^3 \cosh^2(t/R)}{64G}. \quad (4.16)$$

When $\theta_c \rightarrow 0$, it shows that the first correction gives rise to an additional logarithmic divergence unlike the known entanglement entropy. From the solutions obtained perturbatively, θ_c is determined in terms of ϵ

$$\theta_c = \frac{\epsilon^2}{2\theta_v R^2 \cosh^2(t/R)} + \frac{\epsilon^4}{8\theta_v^3 R^4 \cosh^4(t/R)} + \frac{\epsilon^4}{48\theta_v R^4 \cosh^2(t/R)} - \frac{\epsilon^4}{4\theta_v R^4 \cosh^2(t/R)} \log \frac{2\theta_v R \cosh(t/R)}{\epsilon}$$

$$+ O(\epsilon^6). \quad (4.17)$$

Using this relation, the resulting entanglement entropy leads to

$$S_E = \frac{R \mathcal{A}_2(t)}{8G} - \frac{\Omega_2 R^3}{16G} \log \frac{4\mathcal{A}_2(t)}{\Omega_2 R^2} - \frac{\Omega_2 R^3}{16G} + \frac{\theta_v^2 \Omega_2 R^3 \cosh^2(t/R)}{G} \left(\frac{1}{6} - \frac{1}{16} \log \frac{4\mathcal{A}_2(t)}{\Omega_2 R^2} \right), \quad (4.18)$$

where the area of cosmic event horizon is given by

$$\mathcal{A}_2(t) = \frac{\theta_v^2 R^4 \Omega_2 \cosh^2(t/R)}{\epsilon^2}. \quad (4.19)$$

Replacing t and ϵ by τ and H by using Eq. (4.2), θ_v and the area of cosmic event horizon in the late inflation era ($H\tau \gg 1$) are approximated by

$$\theta_v \approx 2e^{-H\tau}, \quad \text{and} \quad \mathcal{A}_2(\tau) \approx \frac{\Omega_2}{H^2}. \quad (4.20)$$

As a result, the entanglement entropy of the visible universe in the late inflation era leads to the following expression

$$S_E \approx \frac{\Omega_2 R}{8GH^2} - \frac{\Omega_2 R^3}{4G} \log \frac{2}{RH} + \frac{5\Omega_2 R^3}{48G}. \quad (4.21)$$

This result shows that the entanglement entropy of the visible universe in the late inflation era is time-independent and determined by the Hubble constant and the area of cosmic event horizon. This is because cosmic event horizon remains as a constant in the late inflation era. The change of the entanglement entropy during the inflation era is given by

$$\Delta S_E \equiv S_E(\infty) - S_E(0) = -\frac{\Omega_2 R^3}{8G} \log \frac{2}{RH} + \frac{\Omega_2 R^3}{24G}, \quad (4.22)$$

where the result in Eq. (4.9) was used. Since $HR = \epsilon/R \ll 1$, ΔS_E always becomes negative. This indicates that the quantum correlation between the visible and invisible universes decreases with time. In Fig. 7, we plot how the entanglement entropy of the visible universe changes as the cosmological time goes on. As expected by the perturbative and analytic calculation, the entanglement entropy gradually decreases and finally approaches a constant value after an infinite time.

5 Discussion

In this work, we have studied the quantum entanglement entropy of the expanding system and the inflationary universe. In order to take into account the expanding system and universe holographically, we considered an AdS space whose boundary is given by a dS space. Because of the time-dependence of the background geometry, we have taken into

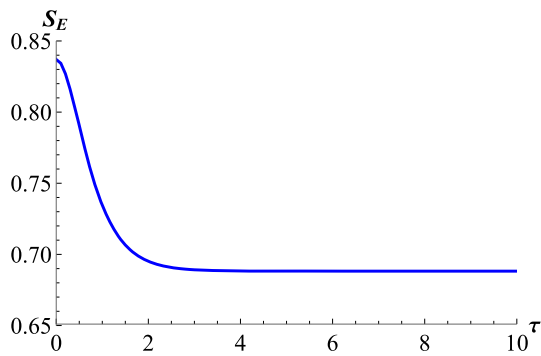


Fig. 7 We take $\epsilon = 1$, $R = 1$ and $G = 1$ for simplicity

account the covariant formulation instead of the RT formula. In general, the minimal surface in the covariant formulation is governed by more complicated equations of the holographic and time directions. We showed that the RT formula with a fixed time can give rise to the leading contribution to the covariant formulation. We also showed that, when we apply the RT formula to the time-dependent geometry, the resulting entanglement entropy has a nontrivial time dependence due to the expansion of the entangling region.

In this work, we took two different subsystems. In Sect. 3, we considered an expanding system where the subsystem size is determined by the parameter θ_o . In this case, since the volume of the boundary space increases with the cosmological time, the subsystem size also increases. In the early time era, we found that the entanglement entropy of an expanding system increases by t^2 regardless of the dimensionality of the system. In the late time era, when the d -dimensional boundary space expands by $e^{H\tau}$, the increase of the entanglement entropy is given by $e^{(d-2)\tau}$ which is proportional to the area of the entangling surface. This fact, intriguingly, indicates that the leading entanglement entropy even in the expanding universe satisfies the area law.

For a dS space discussed in Sect. 4, there is an important length scale called cosmic event horizon. If an observer is at the center of the subsystem, he cannot see the outside of cosmic event horizon even after the infinite time evolution. In other words, the observer at the center of the subsystem can never get any information from the outside of cosmic event horizon. This is similar to the black hole case. From the quantum information viewpoint, there may exist a nonvanishing quantum correlation between two classically disconnected regions. In this case, the cosmic event horizon like the black hole horizon plays a role of the entangling surface dividing a total system into two subsystems. In the present model, the cosmic event horizon starts with $\theta_v = \pi/2$ at $\tau = 0$ and eventually approaches $\theta_v = 0$ at $\tau = \infty$ with a fixed $\theta_v e^{H\tau}$. In the late inflation era, the cosmic event horizon is given by the inverse of the Hubble constant, $d(\infty) = 1/H$. We showed that the entanglement entropy of the visible uni-

verse in the inflationary cosmology decreases continuously with time and that it finally approaches a finite value after an infinite time.

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