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Brief paper

Pose localization of leader–follower networks with direction measurements☆

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ABSTRACT

A distributed pose localization framework based on direction measurements is proposed for a type of *leader–follower* multi-agent system in $\mathbb{R}^3$. The novelty of the proposed localization method lies in the elimination of the need for using distance measurements and relative orientation measurements for the network pose localization problem. In particular, a network localization scheme is developed based directly on the measured directions between an agent and its neighboring agents in the network. The proposed position and orientation localization algorithms are implemented through differential equations which simultaneously compute poses of all followers by using locally measured directional vectors and angular velocities, and actual pose knowledge of some leader agents, allowing some tracking of time-varying orientations. Further, we establish almost global asymptotic convergence of the estimated positions and orientations of the agents to the actual poses in the stationary case.
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1. Introduction

Networked cooperative pose localization tackles the determination of positions and orientations of a networked set of agents in an underlying three dimensional space thorough various interagent measurements. It may well be done in order to perform further coordination control or distributed estimation tasks (Oh, Park, & Ahn, 2015; Zhao & Zelazo, 2019). Distances and directions are the two most commonly used measurements that are widely used in position localization literature (Aspnes et al., 2006; Mao, Fidan, & Anderson, 2007). In a three dimensional ambient space, direction is characterized by a unit length vector, and this can often be obtained by visual imaging, see Ma, Soatto, Kosecka, and Sastry (2004). However, in three-dimensional space, additional relative orientation measurements[1] between neighboring agents are often required for estimating orientations (as opposed to positions) of the agents in a network, a process which is called orientation localization (Piovan, Shames, Fidan, Bullo, & Anderson, 2013; Tron & Vidal, 2014). Nevertheless, there are not many works that study simultaneous localization of positions and orientations, a process which is called pose localization, in a distributed setup. Motivated by these facts, this work attempts to provide a *distributed pose localization framework* for a type of leader–follower networks based on direction measurements and pose knowledge of some leader agents.

For a two-dimensional (2-D) ambient space, network localization laws using angles of arrival between triplets of nodes are proposed in Zhu, Huang, and Jiang (2008) and an orientation localization method utilizing orientation knowledge of a few nodes is presented in Rong and Sitichiu (2006). The authors in Piovan et al. (2013) further proposed a least-squares optimization problem to achieve orientation localization by exploiting kinematic relationships among the orientations of nodes. A least-squares algorithm for position localization using bearing-only information is proposed in Bishop and Shames (2011). In 3-dimensional space (3-D), it is often required that relative orientation measurements are available for estimating the orientations of the agents. For example, some necessary and sufficient conditions

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[1] A relative orientation is effectively the rotation matrix linking a local coordinate frame of one agent to the local coordinate frame of another agent. It is often estimated by vision-based techniques, e.g. by processing images (of a common scene) captured by the agents and establishing the feature correspondences (Tron & Vidal, 2014).

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are provided for orientation localizability of triangular sensing networks of relative orientation measurements in Piovan et al. (2013), without providing a distributed orientation localization law. Network localization schemes using relative poses (relative orientations and relative positions), which are measured by a vision-based technique, are investigated in Thunberg, Bernard, and Gonçalves (2017), Tron and Vidal (2014). The estimation of relative poses, however, generally requires the agents to have views of a common scene. By using relative orientation measurements, our recent works in Tran, Ahn, and Anderson (2018) and Tran, Trinh, Zelazo, Mukherjee, and Ahn (2019) propose distributed orientation estimation laws which guarantee almost global convergence of the estimated orientations up to a common orientation. Zhao and Zelazo (2016) proposes a direction-only position localization law for *bearing rigid* networks with two anchor nodes. However, Zhao and Zelazo (2016) further assumes that the agents know their actual orientations. There is no framework for direction-only network localization and formation control in 3-D when agents lack knowledge of a global frame.

The orientation localization problem is challenging and requires sophisticated estimation algorithms. In 2-D, it is straightforward to see how two neighboring agents observing each other might determine a common view of their relative orientation (i.e., a scalar angle), within an unknown constant rotation common to both, see e.g. Oh and Ahn (2014), as is now described. Each agent maintains a (possibly body-fixed) coordinate frame and measures the orientation angle of its neighboring agent (assuming direction sensing technology). In any common frame, the measured angles (of the two neighboring agents) must differ by precisely π radians. Hence a rotation of the coordinate axes of one agent can be made to ensure that after rotation, the angle difference is compensated. For an n agent network, one has to put together in a distributed fashion a collection of such calculations.

How to do something like this in a 3-dimensional ambient space is less clear. For example, with only a pair of direction measurements between two neighboring agents i and j ($\mathbf{b}_{ij}^i, \mathbf{b}_{ji}^j$) $\in \mathbb{R}^3 \times \mathbb{R}^3$ (see Fig. 1(b)), it is insufficient for the agents i and j to determine their *relative orientation*, i.e., $\mathbf{R}_{ij} \triangleq \mathbf{R}_i^\top \mathbf{R}_j \in SO(3)$, where $\mathbf{R}_i$ and $\mathbf{R}_j \in SO(3)$ are the orientation matrices of agents i and j, respectively, due to the ambiguity of the rotation along the common direction vector, $\mathbf{b}_{ij}$. This difficulty can be overcome by examining additional direction constraints of each of the two agents to a third agent k that they both observe. Indeed, as shown in Tran et al. (2018), by exploiting the triangle sensing network agents i and j can compute $\mathbf{R}_{ij}$. The orientations of all agents then can be computed up to a common orientation bias by using a consensus protocol (Tran et al., 2018). This method, however, relies on the existence of triangle networks and requires predefinition of a complicated computation sequence.

This paper proposes a distributed pose localization scheme for a type of leader–follower network that uses continuous-time directional vectors and two or more anchor agents which know their absolute poses. A distributed orientation localization protocol in $SO(3)$ that estimates orientations of all followers is proposed. Under the proposed orientation localization protocol, estimated orientations converge to the true orientations of agents almost globally and asymptotically. By using the estimates of orientations and direction measurements, we investigate a position localization law for the *leader–follower* network. Under the proposed position localization law, positions of all followers are also globally and asymptotically determined. The proposed network pose localization scheme can work exclusively with inter-agent directional vectors.

The rest of this paper is organized as follows. Section 2 presents some preliminaries and the problem formulation. The orientation localization problem is studied in Section 3. We propose a position localization law in Section 4. Finally, Section 5 concludes this paper.

2. Preliminaries and problem formulation

Notation. The dot product and cross product are denoted by $\cdot$ and $\times$, respectively. The symbol Σ represents a global coordinate frame and the symbol ${}^k\Sigma$ with the superscript index k denotes the kth local coordinate frame. Let $\mathbf{1}_n \in \mathbb{R}^n$ be the vector of all ones, and $\mathbf{I}_3$ the 3×3 identity matrix. The trace of a matrix is denoted by $\mathrm{tr}(\cdot)$. The set of rotation matrices in $\mathbb{R}^3$ is denoted by $SO(3) = \{\mathbf{Q} \in \mathbb{R}^{3\times 3} \mid \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}_3, \det(\mathbf{Q}) = 1\}$. The set of real matrices with orthonormal column vectors is $O(3)$. The orthogonal projection matrix associated with a nonzero vector $\mathbf{x} \in \mathbb{R}^3$ is defined as

$$\mathbf{P}_\mathbf{x} = \mathbf{I}_3 - \frac{\mathbf{x}}{\|\mathbf{x}\|}\frac{\mathbf{x}^\top}{\|\mathbf{x}\|} \in \mathbb{R}^{3\times 3}. \tag{1}$$

It can be verified that $\mathbf{P}_\mathbf{x}$ is positive semidefinite and idempotent. Moreover, $\mathbf{P}_\mathbf{x}$ has the nullspace $\mathrm{null}(\mathbf{P}_\mathbf{x}) = \mathrm{span}\{\mathbf{x}\}$ and the eigenvalue set $\{0, 1, 1\}$ (Zhao & Zelazo, 2019). The space of 3×3 skew-symmetric matrices is denoted by $\mathfrak{so}(3) := \{\mathbf{A} \in \mathbb{R}^{3\times 3} | \mathbf{A}^\top = -\mathbf{A}\}$. For any $\omega \in \mathbb{R}^3$, the *hat* map $(\cdot)^\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined such that $\omega \times \mathbf{v} = \omega^\wedge \mathbf{v}, \forall \mathbf{v} \in \mathbb{R}^3$. The *vee* map is the inverse of the *hat* map and defined as $(\cdot)^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$ (Bullo & Lewis, 2005). The *exponential map exp* : $\mathfrak{so}(3) \to SO(3)$ is *surjective* and $T_\mathbf{R}SO(3) = \{\mathbf{R}\eta^\wedge : \eta^\wedge \in \mathfrak{so}(3)\}$ denotes the tangent space at a point $\mathbf{R} \in SO(3)$.

2.1. Directional vector and orientation of agent

Consider a network of n nodes in $\mathbb{R}^3$. Each node corresponds to an agent, and an agent is defined by the position of its centroid and the orientation of a body-fixed coordinate frame ${}^i\Sigma$ relative to a global frame Σ. In the sequel, the position of an agent will be taken to be the position of its centroid. Let $\mathbf{p}_i$ and $\mathbf{p}_i^i \in \mathbb{R}^3$ be the position of agent i expressed in Σ and ${}^i\Sigma$, respectively. We define the unit directional vector (expressed in Σ) pointing from agent i toward its neighbor j along the direction of $\mathbf{p}_{ij} := \mathbf{p}_j - \mathbf{p}_i$ as $\mathbf{b}_{ij} \triangleq \mathbf{p}_{ij}/\|\mathbf{p}_{ij}\|$. The directional vector with the reverse direction is $\mathbf{b}_{ji} = -\mathbf{b}_{ij}$. The directional vector $\mathbf{b}_{ij}$ measured locally in ${}^i\Sigma$ is denoted as $\mathbf{b}_{ij}^i$. The orientation of agent i in $\mathbb{R}^3$ can be characterized by a square, orthogonal matrix $\mathbf{R}_i \in SO(3)$. The pair $(\mathbf{R}_i, \mathbf{p}_i) \in SE(3)$ characterizes the *pose* of agent i in the global Cartesian space.

2.2. Graph theory

An interaction graph characterizing an interaction topology of a multi-agent network is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where, $\mathcal{V} = \{1, \ldots, n\}$ denotes the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges of $\mathcal{G}$. An edge is defined by the ordered pair $e_k = (i, j), k = 1, \ldots, m, m = |\mathcal{E}|$. The graph $\mathcal{G}$ is said to be undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$, i.e. if j is a neighbor of i, then i is also a neighbor of j. If the graph $\mathcal{G}$ is directed, $(i, j) \in \mathcal{E}$ does not necessarily imply $(j, i) \in \mathcal{E}$. The set of neighboring agents of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

2.3. Problem formulation

Consider a leader–follower network in $\mathbb{R}^3$ with at least two non-collocated leader agents 1 and 2 which know their actual poses (position and orientation in a global coordinate frame). The leader–follower network studied in this work is defined as follows (See also Fig. 1(a)).

Definition 1 (*Twin-Leader–Follower Network*)**.** A twin-leader–follower network is a directed network in which agents are ordered such that (a) all leader agents appear first, there are two (or more) leaders 1 and 2 which know their absolute poses $(\mathbf{R}_1, \mathbf{p}_1)$ and $(\mathbf{R}_2, \mathbf{p}_2)$, respectively (b) a follower agent i, $3 \le i \le n$, has at least two neighboring agents j's in the set $\{1, \ldots, i-1\}$, i.e., $|\mathcal{N}_i| \ge 2$, where $\mathcal{N}_i$ denotes the set of neighboring agents

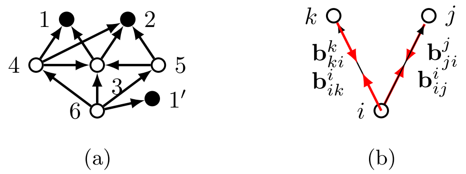

Fig. 1. (a) A *twin-leader–follower* network: leader nodes 1, 2, and 1′, the first follower 3. (b) Agent i measures $(\mathbf{b}^i_{ij}, \mathbf{b}^i_{ik})$, $j, k \in \mathcal{N}_i$ and receives $(\hat{\mathbf{R}}_j\mathbf{b}^j_{ji}, \hat{\mathbf{R}}_k\mathbf{b}^k_{ki})$ from j and k.

of i. Agent i knows the direction $\mathbf{b}^i_{ij}$ to the neighbor j, while its neighbor knows the direction $\mathbf{b}^j_{ji}$.

Note importantly that with only one leader, it is impossible to compute the actual agent poses due to the translational and scale ambiguities in networks with direction-only measurements (Trinh, et al., 2019; Zhao & Zelazo, 2019). Further, without pose knowledge of the two leaders, the two leaders can arbitrarily select the translation, rotation, and scale factors of the pose estimation (Tran, Anderson, & Ahn, 2019, Remark 3). We remark that the first listed nonleader agent is known as a first follower and any leader agents beyond the first two are known as redundant leaders. To streamline nomenclature, we number the agents as $\{1, 2, 1', 2', \ldots, 3, 4, \ldots, n\}$, where the follower agents are $3, 4, \ldots, n$; also $\mathcal{V}_l = \{1, 2, 1', 2' \ldots\}$, where $1', 2' \ldots$ are redundant leaders, and $\mathcal{V}_f = \{3, 4, \ldots, n\}$ will denote the sets of leader and follower agents, respectively.

Each agent $i \in \mathcal{V}_f$ in the network aims to estimate its actual pose, i.e., $(\mathbf{R}_i, \mathbf{p}_i) \in SO(3) \times \mathbb{R}^3$, based on the direction constraints to its neighboring agents and the actual poses of the leader agents. At each time instant t agent i holds an estimate of its pose, denoted as $(\hat{\mathbf{R}}_i, \hat{\mathbf{p}}_i) \in SO(3) \times \mathbb{R}^3$.

Assumption 1. Agent j estimates its orientation at time t by $\hat{\mathbf{R}}_j$, and transmits the information $\hat{\mathbf{R}}_j\mathbf{b}^j_{ji}$ to agent $i, j \in \mathcal{N}_i$ (see Fig. 1(b)).

We assume that the agents in the network do not translate but they might rotate according to the kinematics

$$\dot{\mathbf{R}}_i = \mathbf{R}_i(\omega^i_i)^\wedge, \text{ for } i \in \mathcal{V},$$

where ω^i is the angular velocity of agent i measured locally in ${}^i\Sigma$. We assume that ω^i_i and its derivative are bounded, i.e., $\|\omega^i_i\| \le \bar{\omega}_i$, $\|\dot{\omega}^i_i\| \le \dot{\bar{\omega}}_i$, for positive constants $\bar{\omega}_i, \dot{\bar{\omega}}_i > 0$, and each agent i can measure ω^i_i without noise. The angular velocity expressed in the global coordinates is $\omega_i = \mathbf{R}_i\omega^i_i$. This kind of system might represent a visual sensor network (Tron & Vidal, 2014) or a system of autonomous agents in a desired formation (Tran, Trinh, Zelazo, Mukherjee, & Ahn, 2019) where the agents might rotate to track objects. Moreover, to secure uniqueness of the localized poses of the agents, we have the following assumption.

Assumption 2. No two agents are collocated and each follower $i \in \mathcal{V}_f$ has at least one pair of neighbors with which it is not collinear.

We first address the problem of calculating the orientation $\hat{\mathbf{R}}_i$ for all follower agents.

Problem 1. Considering a *twin-leader–follower* network of n agents, under Assumptions 1–2, compute $\hat{\mathbf{R}}_i$ for each follower $i \in \mathcal{V}_f$ based on the directional measurements $(\mathbf{b}^i_{ij}, \mathbf{b}^j_{ji})$, estimated orientations of its neighbors $\hat{\mathbf{R}}_j, j \in \mathcal{N}_i$, and the knowledge of the true orientations of the two or more leaders, i.e., $\mathbf{R}_k \in SO(3)$, $k \in \mathcal{V}_l$.

Assuming solvability of Problem 1, the second problem investigated is to determine the locations of agents.

Problem 2. Consider a *twin-leader–follower* network of n nontranslating but possibly rotating agents with at least two leaders. Under Assumptions 1–2, for each follower i, determine its actual position, $\mathbf{p}_i \in \mathbb{R}^3$, based on the estimate $\hat{\mathbf{R}}_i$, the direction constraints $\mathbf{b}^i_{ij}, j \in \mathcal{N}_i$, and absolute positions of some leaders, i.e., $\mathbf{p}_k \in \mathbb{R}^3$, $k \in \mathcal{V}_l$.

3. Orientation localization

In this section, we present a differential equation constituting a continuous-time orientation localization law in $SO(3)$ that computes time-varying orientations of agents simultaneously using continuous-time directional vectors to multiple neighboring agents, angular velocity measurements, and actual orientations of some leaders. Further, the equilibrium set of the differential equation is first characterized and almost global asymptotic convergence of the estimated orientations is established.

3.1. Error function and critical points

Consider an agent $i \in \mathcal{V}_f$ which senses the local directions, $\mathbf{b}^i_{ij} \in \mathbb{R}^3$, to its neighboring agents $j \in \mathcal{N}_i$. If $|\mathcal{N}_i| = 2$, the third direction constraint is defined by the normalized cross product of the first two directions, for positive definiteness of $\mathbf{K}_i$ in (3). The objective is to find an estimate, $\hat{\mathbf{R}}_i \in SO(3)$, of the true orientation, $\mathbf{R}_i$, that is a minimum of the following error function

$$\begin{aligned}\Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i) &= 1/2\sum_{j\in\mathcal{N}_i} k_{ij}\|\hat{\mathbf{R}}_i\mathbf{b}^i_{ij} - \mathbf{b}_{ij}\|^2\\ &= \sum_{j\in\mathcal{N}_i} k_{ij}(1 - \hat{\mathbf{R}}_i\mathbf{b}^i_{ij}\cdot\mathbf{b}_{ij}),\end{aligned} \tag{2}$$

which is sum of squared norms of all direction constraint errors. We do not assert that Φ_i can be evaluated from the measurements, but we shall show that it can be minimized from the measurements. In (2), positive constant gains, $k_{ij} \in \mathbb{R}$, are used to impose different weights on error terms in the error function. The above configuration error function is in the form of Wahba's cost function (Wahba, 1965) and used for attitude tracking control (Lee, 2015) or attitude estimation of a rigid body (Izadi & Sanyal, 2016; Mahony, Hamel, & Pflimlin, 2008). In the sequel, we follow techniques similar to those in Bullo and Lewis (2005, Chap. 11), (Lee, 2015) to design our orientation localization law.

Let $\Phi_{ij} := 1 - \hat{\mathbf{R}}_i\mathbf{b}^i_{ij}\cdot\mathbf{b}_{ij} = 1 - \mathrm{tr}(\hat{\mathbf{R}}_i\mathbf{b}^i_{ij}\mathbf{b}_{ij}^\top) = 1 - \mathrm{tr}(\hat{\mathbf{R}}_i\mathbf{R}_i^\top\mathbf{b}_{ij}\mathbf{b}_{ij}^\top)$, where we use the relations $\mathbf{x}^\top\mathbf{y} = \mathrm{tr}(\mathbf{x}\mathbf{y}^\top)$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, and $\mathbf{b}^i_{ij} = \mathbf{R}_i^\top\mathbf{b}_{ij}$. Let $\tilde{\mathbf{Q}}_i \triangleq \hat{\mathbf{R}}_i\mathbf{R}_i^\top$ and hence $\Phi_{ij} = 1 - \mathrm{tr}(\tilde{\mathbf{Q}}_i\mathbf{b}_{ij}\mathbf{b}_{ij}^\top)$. Consider a vector in the tangent space of $SO(3)$ at the point $\hat{\mathbf{R}}_i$ (resp. $\mathbf{R}_i$) as $\delta\hat{\mathbf{R}}_i = \hat{\mathbf{R}}_i\eta_i^\wedge$, $\eta_i \in \mathbb{R}^3$, (resp. $\delta\mathbf{R}_i = \mathbf{R}_i\zeta_i^\wedge$, $\zeta_i \in \mathbb{R}^3$) (Bullo & Lewis, 2005). Then, the following straightforwardly established lemma can be proved (Tran, Anderson, & Ahn, 2019).

Lemma 1. *The derivative of the error function $\Phi(\hat{\mathbf{R}}_i, \mathbf{R}_i)$ with respect to $\hat{\mathbf{R}}_i$ (resp. $\mathbf{R}_i$) along the direction of $\hat{\mathbf{R}}_i\eta_i^\wedge$ (resp. $\mathbf{R}_i\zeta_i^\wedge$) is given by*

$$\mathbf{D}_{\hat{\mathbf{R}}_i}\Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i)\cdot\hat{\mathbf{R}}_i\eta_i^\wedge = \eta_i^\top\sum_{j\in\mathcal{N}_i}\mathbf{e}_{ij},$$

$$\Big(\textit{resp. } \mathbf{D}_{\mathbf{R}_i}\Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i)\cdot\mathbf{R}_i\zeta_i^\wedge = -\zeta_i^\top\sum_{j\in\mathcal{N}_i}\mathbf{e}_{ij}\Big),$$

where $\mathbf{e}_{ij} \triangleq k_{ij}(\hat{\mathbf{R}}_i^\top\mathbf{b}_{ij} \times \mathbf{b}^i_{ij}) \in \mathbb{R}^3$, $j = 1, \ldots, |\mathcal{N}_i|$.

We now study the critical points of $\Phi_i(\hat{\mathbf{R}}_i)$. To proceed, we rewrite the error function as

$$\Phi_i = \sum_{j\in\mathcal{N}_i} k_{ij} - \sum_{j\in\mathcal{N}_i} \mathrm{tr}(k_{ij}\hat{\mathbf{R}}_i\mathbf{R}_i^\top \mathbf{b}_{ij}\mathbf{b}_{ij}^\top) = \sum_{j\in\mathcal{N}_i} k_{ij} - \mathrm{tr}(\tilde{\mathbf{Q}}_i\mathbf{K}_i) \tag{3}$$

where $\mathbf{K}_i \triangleq \sum_{j\in\mathcal{N}_i} k_{ij}\mathbf{b}_{ij}(\mathbf{b}_{ij})^\top \in \mathbb{R}^{3\times 3}$, which can be shown to have distinct eigenvalues for almost all positive scalars k_{ij} by using the fact that the zero set of the discriminant of the characteristic equation of $\mathbf{K}_i$, which is a polynomial of the real entries of $\mathbf{K}_i$, is a set of measure zero. Since Range$\left(k_{ij}\mathbf{b}_{ij}(\mathbf{b}_{ij})^\top\right) = \mathrm{span}\{\mathbf{b}_{ij}\}$, it can be verified that $\mathbf{K}_i$ is positive definite if and only if $\{\mathbf{b}_{ij}\}_{j\in\mathcal{N}_i}$ are non-coplanar. Thus, $\mathbf{K}_i$ can be decomposed as $\mathbf{K}_i = \mathbf{UGU}^\top$ where $\mathbf{G} = \mathrm{diag}\{\lambda_k(\mathbf{K}_i)\}, \lambda_k(\mathbf{K}_i) > 0,\ k = 1, 2, 3,$ and $\mathbf{U} \in O(3)$. Also note that $\mathrm{tr}(\mathbf{G}) = \mathrm{tr}(\mathbf{K}_i) = \mathrm{tr}(\sum_{j\in\mathcal{N}_i} k_{ij}\mathbf{b}_{ij}\mathbf{b}_{ij}^\top) = \sum_{j\in\mathcal{N}_i} k_{ij}\mathbf{b}_{ij}^\top\mathbf{b}_{ij} = \sum_{j\in\mathcal{N}_i} k_{ij}$. Consequently, one has $\Phi_i = \mathrm{tr}(\mathbf{G}) - \mathrm{tr}(\tilde{\mathbf{Q}}_i\mathbf{UGU}^\top) = \mathrm{tr}(\mathbf{G}(\mathbf{I}_3 - \mathbf{U}^\top\tilde{\mathbf{Q}}_i\mathbf{U}))$, whose critical points are given as follows.

Lemma 2 (Bullo & Lewis, 2005, *Prop. 11.31*)**.** *Let $\mathbf{G}$ be a diagonal matrix with distinct positive entries and $\mathbf{U} \in O(3)$. Then, $\Phi_i(\tilde{\mathbf{Q}}_i) = tr(\mathbf{G}(\mathbf{I}_3 - \mathbf{U}^\top\tilde{\mathbf{Q}}_i\mathbf{U}))$ has four critical points given by $\tilde{\mathbf{Q}}_i \in \{\mathbf{I}_3, \mathbf{UD}_1\mathbf{U}^\top, \mathbf{UD}_2\mathbf{U}^\top, \mathbf{UD}_3\mathbf{U}^\top\}$, where $\mathbf{D}_i = 2[\mathbf{I}_3]_i[\mathbf{I}_3]_i^\top - \mathbf{I}_3$ and $[\mathbf{I}_3]_i$ is the ith column vector of $\mathbf{I}_3$.*

Those critical points are clearly isolated in which $\tilde{\mathbf{Q}}_i = \hat{\mathbf{R}}_i\mathbf{R}_i^\top = \mathbf{I}_3$ is the desired point and $\mathrm{tr}(\tilde{\mathbf{Q}}_i) = -1$ for the three undesired points.

3.2. Orientation localization law

We now propose orientation localization law for each follower agent i as

$$\dot{\hat{\mathbf{R}}}_i = \hat{\mathbf{R}}_i\Omega_i^\wedge, \tag{4}$$

where the control vector $\Omega_i \in \mathbb{R}^3$ will be designed later and $\hat{\mathbf{R}}_i(0)$ is initialized arbitrarily in $SO(3)$. Let $\tilde{\Omega}_i \triangleq \omega_i^i - \Omega_i$; we have the following lemma, which can be proved using techniques similar to those in Lee (2015, Prop. 1).

Lemma 3. *The vector, $\mathbf{e}_i \triangleq \sum_{j\in\mathcal{N}_i} \mathbf{e}_{ij}$, and error function, Φ_i, in (2) satisfy the following properties*

(i) $\|\dot{\mathbf{e}}_i\| \le \sum_{j\in\mathcal{N}_i} k_{ij}\|\tilde{\Omega}_i\| + \bar{\omega}_i\|\mathbf{e}_i\|$, *where the positive constant* $\bar{\omega}_i > 0$ *satisfies* $\|\omega_i\| \le \bar{\omega}_i$,

(ii) $\dot{\Phi}_i(\hat{\mathbf{R}}_i, \mathbf{R}_i) = -\tilde{\Omega}_i \cdot \mathbf{e}_i$,

(iii) There exist constants $\sigma_i, \gamma_i > 0$ *such that* $\sigma_i\|\mathbf{e}_i\|^2 \le \Phi_i(\hat{\mathbf{R}}_i, \mathbf{R}_i) \le \gamma_i\|\mathbf{e}_i\|^2$, *where the upper bound holds when* $\Phi_i < 2\min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}$, $(\lambda_k = \lambda(\mathbf{K}_i), k = 1, 2, 3)$.

The control vector $\Omega_i = \omega_i^i - \tilde{\Omega}_i$, where $\tilde{\Omega}_i \in \mathbb{R}^3$ is designed via

$$\dot{\tilde{\Omega}}_i = -k_\omega\tilde{\Omega}_i + \sum_{j\in\mathcal{N}_i} k_{ij}(\hat{\mathbf{R}}_i^\top\hat{\mathbf{R}}_j\mathbf{b}_{ij}^j \times \mathbf{b}_{ij}^i), \tag{5}$$

where $k_\omega > 0$ is a positive constant. The orientation localization law (4)–(5) is distributed since only directional vectors, i.e., $\mathbf{b}_{ij}^i$, and information communicated from neighboring agents, i.e., the estimate of direction in global coordinates, $\hat{\mathbf{R}}_j\mathbf{b}_{ij}^j$, are utilized. Since the right hand side of (5) is linear in $\tilde{\Omega}_i$ and the second term is bounded, $\tilde{\Omega}_i$ is uniformly continuous in t.

3.3. Stability and convergence analysis

We rewrite (5) as

$$\begin{aligned}\dot{\tilde{\Omega}}_i &= -k_\omega\tilde{\Omega}_i + \sum_{j\in\mathcal{N}_i} k_{ij}(\hat{\mathbf{R}}_i^\top\mathbf{b}_{ij} \times \mathbf{b}_{ij}^i \\ &\quad + \hat{\mathbf{R}}_i^\top(\hat{\mathbf{R}}_j - \mathbf{R}_j)\mathbf{b}_{ij}^j \times \mathbf{b}_{ij}^i) \\ &= -k_\omega\tilde{\Omega}_i + \mathbf{e}_i + \mathbf{h}_i(\hat{\mathbf{R}}_j, t),\end{aligned} \tag{6}$$

where $\mathbf{h}_i(\hat{\mathbf{R}}_j, t) = \sum_{j\in\mathcal{N}_i} k_{ij}(\hat{\mathbf{R}}_i^\top(\hat{\mathbf{R}}_j - \mathbf{R}_j)\mathbf{b}_{ij}^j \times \mathbf{b}_{ij}^i)$. Due to the cascade structure of the leader–follower system we prove the almost global convergence of the estimated orientations using an induction argument.

3.3.1. The first follower

For the first follower, i.e., agent 3, we have $\mathbf{h}_3 = \mathbf{0}$. Thus,

$$\dot{\hat{\mathbf{R}}}_3 = \hat{\mathbf{R}}_3(\omega_3^3 - \tilde{\Omega}_3)^\wedge,\ \ \dot{\tilde{\Omega}}_3 = -k_\omega\tilde{\Omega}_3 + \mathbf{e}_3. \tag{7}$$

Theorem 1. *Suppose that Assumptions 1–2 hold. Then, under the orientation localization law (7), we have:*

(i) The equilibrium points of (7) are given as $\{(\tilde{\mathbf{Q}}_3, \tilde{\Omega}_3) \mid \tilde{\mathbf{Q}}_3 \in \{\mathbf{I}_3, \mathbf{UD}_1\mathbf{U}^\top, \mathbf{UD}_2\mathbf{U}^\top, \mathbf{UD}_3\mathbf{U}^\top\}, \tilde{\Omega}_3 = \mathbf{0}\}$, *where* $\mathbf{D}_i$ *and* $\mathbf{U}$ *are defined in Lemma 2.*

(ii) The desired equilibrium, $(\tilde{\mathbf{Q}}_3 = \mathbf{I}_3, \tilde{\Omega}_3 = \mathbf{0})$ *is almost globally asymptotically stable (aGAS),* $\tilde{\mathbf{Q}}_3 = \mathbf{I}_3$ *is the global minimum of* $\Phi_3(\tilde{\mathbf{Q}}_3)$ *and the three undesired equilibria are unstable.*

Proof. Consider the Lyapunov function

$$V_3 = 1/2\tilde{\Omega}_3^2 + \Phi_3(\hat{\mathbf{R}}_3, \mathbf{R}_3) - k_V\tilde{\Omega}_3 \cdot \mathbf{e}_3, \tag{8}$$

for a constant $k_V > 0$. Following Lemma 3(iii), we can show that

$$V_3 \ge 1/2\mathbf{z}_3^\top \begin{bmatrix} 1 & -k_V \\ -k_V & 2\sigma_3 \end{bmatrix} \mathbf{z}_3,$$

where $\mathbf{z}_3^\top = \left[\|\tilde{\Omega}_3\|, \|\mathbf{e}_3\|\right]$. It follows that $V_3 \ge 0$ if and only if $k_V < \sqrt{2\sigma_3}$. It can be shown that the time derivative of V_3 along the trajectory of (7) is negative definite, i.e., $\dot{V}_3 \le -1/2\mathbf{z}_3^\top\mathbf{M}_3\mathbf{z}_3$, for a positive definite matrix $\mathbf{M}_3$, if k_V is sufficiently small. This bounds $V_3(t) \le V_3(0)$ and consequently $\tilde{\Omega}_3$ is bounded. A direct calculation of $\ddot{V}_3$ shows that $\ddot{V}_3$ is bounded due to the boundedness of $\dot{\tilde{\Omega}}_3$ and $\dot{\mathbf{e}}_3$ (Lemma 3(i)). As a result, $\tilde{\Omega}_3(t) \to \mathbf{0}, \mathbf{e}_3(t) \to \mathbf{0}$ as $t \to \infty$ according to Barbalat's lemma. Consequently, the equilibrium points of (7) satisfy $\tilde{\Omega}_3 = \mathbf{0}$ and $\tilde{\mathbf{Q}}_3$ comprises critical points of $\Phi_3(\tilde{\mathbf{Q}}_3)$ (since $\dot{\Phi}_3(\tilde{\mathbf{Q}}_3) = -\tilde{\Omega}_3 \cdot \mathbf{e}_3 \to 0$ by Lemma 2); hence (i) is proved.

We show (ii) as follows. Note that the Hessian of $\Phi_3(\tilde{\mathbf{Q}}_3)$ at the desired equilibrium $\tilde{\mathbf{Q}}_3 = \mathbf{I}_3$ is positive definite (Bullo & Lewis, 2005, Prop. 11.31). Consequently, the equilibrium $(\hat{\mathbf{R}}_3 = \mathbf{R}_3, \tilde{\Omega}_3 = \mathbf{0})$ is (locally) asymptotically stable. One can also show that the three undesired equilibria of (7) are unstable and have $\tilde{\mathbf{Q}}_3$ that are either maximum or strict saddle points of Φ_3. Consequently, the desired equilibrium is aGAS except on a set of measure zero in $SO(3)$ which contains the stable manifolds of the undesired equilibrium points. □

It follows from the above theorem that $\hat{\mathbf{R}}_3 \to \mathbf{R}_3$ almost globally asymptotically as $t \to \infty$. For induction, we now suppose that the corresponding result holds for agents $k-1, k-1 \ge 3$, i.e., $\hat{\mathbf{R}}_{k-1} \to \mathbf{R}_{k-1}$ as $t \to \infty$ almost globally. We show that it is also true for the agent k as follows.

3.3.2. Follower k

Using (4) and (6), we have

$$\dot{\hat{\mathbf{R}}}_k = \hat{\mathbf{R}}_k(\omega_k^k - \tilde{\Omega}_k)^\wedge,\ \ \dot{\tilde{\Omega}}_k = -k_\omega\tilde{\Omega}_k + \mathbf{e}_k + \mathbf{h}_k(t), \tag{9}$$

where $\mathbf{h}_k(t) = \sum_{j\in\mathcal{N}_k} k_{kj}(\hat{\mathbf{R}}_k^\top(\hat{\mathbf{R}}_j - \mathbf{R}_j)\mathbf{b}_{kj}^j \times \mathbf{b}_{kj}^k)$ which is clearly bounded and converges to zero asymptotically since $\hat{\mathbf{R}}_j \to \mathbf{R}_j$, $\forall j = 1, \ldots, k-1$. Note that $\mathbf{h}_k(t)$ can be considered as an additive input to the nominal system

$$\dot{\hat{\mathbf{R}}}_k = \hat{\mathbf{R}}_k(\omega_k^k - \tilde{\Omega}_k)^\wedge,\ \ \dot{\tilde{\Omega}}_k = -k_\omega\tilde{\Omega}_k + \mathbf{e}_k. \tag{10}$$

It is noted that the above system is in a similar form to (7) and hence the following result follows directly.

Lemma 4. *Consider the nominal system (10) under the Assumptions 1–2, then:*

(i) The equilibrium points of (10) are given as $\{(\tilde{\mathbf{Q}}_k, \tilde{\Omega}_k) | \tilde{\mathbf{Q}}_k \in \{\mathbf{I}_3, \mathbf{U}\mathbf{D}_1\mathbf{U}^\top, \mathbf{U}\mathbf{D}_2\mathbf{U}^\top, \mathbf{U}\mathbf{D}_3\mathbf{U}^\top\}, \tilde{\Omega}_k = \mathbf{0}\}$*, where* $\mathbf{D}_i$ *and* $\mathbf{U}$ *are defined in Lemma 2.*

(ii) The desired equilibrium, $(\tilde{\mathbf{Q}}_k = \mathbf{I}_3, \tilde{\Omega}_k = \mathbf{0})$ *is aGAS while the three undesired equilibria are unstable.*

The perturbed system (9) is linear in $\tilde{\Omega}_k$ and $\mathbf{e}_k + \mathbf{h}_k$ is bounded. Thus $\tilde{\Omega}_k$ is bounded. Define the set $\mathcal{S}_k \triangleq \{\tilde{\mathbf{Q}}_k | \Phi_k(\tilde{\mathbf{Q}}_k) < \phi_k\}$, where $\phi_k = 2\min\{\lambda_1 + \lambda_2, \lambda_1 + \lambda_3, \lambda_2 + \lambda_3\}$, $\{\lambda_i\}_{i=1,2,3} = \lambda(\mathbf{K}_k)$, or, i.e., the minimum value of Φ_k evaluated at the three undesired critical points.

Lemma 5. *Suppose that Assumptions 1–2 hold. The perturbed system (9) is input-to-state stable (ISS) with respect to* $\mathbf{h}_k(t)$*.*

Proof. Consider the Lyapunov function $V_k = 1/2\tilde{\Omega}_k^2 + \Phi_k - k_V\tilde{\Omega}_k \cdot \mathbf{e}_k$. Then, following from Lemma 3(iii), in $\mathcal{S}_k$ one has

$$1/2\mathbf{z}_k^\top \mathbf{A}_k \mathbf{z}_k \le V_k \le 1/2\mathbf{z}_k^\top \mathbf{B}_k \mathbf{z}_k, \tag{11}$$

where $\mathbf{z}_k^\top = \left[\|\tilde{\Omega}_k\|, \|\mathbf{e}_k\|\right]$ and $\mathbf{A}_k = \begin{bmatrix} 1 & -k_V \\ -k_V & 2\sigma_k \end{bmatrix}$, $\mathbf{B}_k = \begin{bmatrix} 1 & -k_V \\ -k_V & 2\gamma_k \end{bmatrix}$. It can be shown that the time derivative of V_k along the trajectory of (9) satisfies $\dot{V}_k \le -1/2\mathbf{z}_k^\top \mathbf{C}_k \mathbf{z}_k + d\|\mathbf{h}_k\|$, where $d = \sup_t(\|\tilde{\Omega}_k - k_V\mathbf{e}_k\|)$ and $\mathbf{C}_k \in \mathbb{R}^{2\times 2}$ is a positive definite matrix (and so are $\mathbf{A}_k$ and $\mathbf{B}_k$) if k_V is sufficiently small. Therefore, it follows from (11) and the boundedness of $\|\tilde{\Omega}_k\|$ we have that

$$\dot{V}_k \le -\lambda_{\min}(\mathbf{C}_k)/\lambda_{\max}(\mathbf{B}_k)V_k + d\|\mathbf{h}_k\| \tag{12}$$

which shows ultimate boundedness of the system (9) and input-to-state stability of the system (9) w.r.t. $\mathbf{h}_k(t)$ according to Angeli and Praly (2011, Prop. 3).

It follows from (12) that $\dot{V}_k < 0$ if

$$V_k > (\lambda_{\max}(\mathbf{B}_k)/\lambda_{\min}(\mathbf{C}_k))d\|\mathbf{h}_k\| =: \epsilon_1 \tag{13}$$

Define the sublevel set $\mathcal{L}_\epsilon := \{(\tilde{\mathbf{Q}}_k, \tilde{\Omega}_k) \in SO(3) \times \mathbb{R}^3 | V_k \le \epsilon\}$. Then, $\mathcal{L}_{\epsilon_1}$ is a positive invariant set. Since $\|\mathbf{h}_k(t)\|$ tends to zero as $t \to \infty$, the same is true for V_k. To guarantee that $\tilde{\mathbf{Q}}_k \in \mathcal{S}_k = \{\tilde{\mathbf{Q}}_k | \Phi_k(\tilde{\mathbf{Q}}_k) < \phi_k\}$ we consider $V_k < \phi_k\lambda_{\min}(\mathbf{A}_k)/2\gamma_k =: \epsilon_2$. Then, following Lemma 3(iii) and (11), one has $\Phi_k \le \gamma_k\|\mathbf{e}_k\|^2 \le \gamma_k\|\mathbf{z}_k\|^2 \le 2\gamma_k V_k/\lambda_{\min}(\mathbf{A}_k) < \phi_k$. Consequently, any trajectory initializes in or enters $\mathcal{L}_{\epsilon_2}$ will converge to $\mathcal{L}_{\epsilon_1}$, and eventually reach $(\tilde{\mathbf{Q}}_k = \mathbf{I}_3, \tilde{\Omega}_k = \mathbf{0})$ as $t \to \infty$. □

Theorem 2. *Suppose that Assumptions 1–2 hold. Then, the desired equilibrium point,* $(\hat{\mathbf{R}}_k = \mathbf{R}_k, \tilde{\Omega}_k = \mathbf{0})$*, of the system (9) is almost globally asymptotically stable.*

Proof. First, the desired equilibrium point $(\hat{\mathbf{R}}_k = \mathbf{R}_k, \tilde{\Omega}_k = \mathbf{0})$ of the unforced system (10) is aGAS (Lemma 4). The other undesired equilibria are isolated and unstable. The perturbed system (9) satisfies ultimate boundedness and is ISS w.r.t. $\mathbf{h}_k$ (Lemma 5). The input $\mathbf{h}_k(t)$ is bounded and vanishes asymptotically as $t \to \infty$. Consequently, the desired equilibrium point of the system (9) is aGAS (Angeli, 2004; Angeli & Praly, 2011). □

It follows that $\hat{\mathbf{R}}_k(t) \to \mathbf{R}_k$ almost globally asymptotically as $t \to \infty$. Finally, by invoking mathematical induction, the above theorem holds for all $k = 3, \ldots, n$.

Corollary 1. *Suppose direction measurements include bounded additive measurement noise. Then for a sufficiently small bound,* $(\tilde{\mathbf{Q}}_k, \tilde{\Omega}_k)$ *converges to a neighborhood of the desired equilibrium* $(\mathbf{I}_3, \mathbf{0})$ *of (9).*

Proof. The proof follows from the ISS of the system (9) w.r.t. input (Lemma 5). In particular, let $\boldsymbol{\delta} \in \mathbb{R}^3$ be the augmented error vector introduced by the direction measurement errors in (9) i.e.,

$$\dot{\hat{\mathbf{R}}}_k = \hat{\mathbf{R}}_k(\omega_k^k - \tilde{\Omega}_k)^\wedge, \quad \dot{\tilde{\Omega}}_k = -k_\omega\tilde{\Omega}_k + \mathbf{e}_k + \mathbf{h}_k(t) + \boldsymbol{\delta} \tag{14}$$

Following similar arguments in the proof of Lemma 5 we can show that if $\|\boldsymbol{\delta}\|$ is sufficiently small the trajectory of (14) converges to a neighborhood of the desired equilibrium point as $t \to \infty$, i.e., $\{\|\mathbf{z}_k\|^2 \le 2\lambda_{\max}(\mathbf{B}_k)/(\lambda_{\min}(\mathbf{C}_k)\lambda_{\min}(\mathbf{A}_k))d\|\boldsymbol{\delta}\|\}$, which completes the proof. □

4. Position localization

This section investigates position localization using locally measured directions $\mathbf{b}_{ij}^i$, the estimated orientation $\hat{\mathbf{R}}_i$ of agent i and the absolute positions of some leaders. For this, we first study the uniqueness of the target positions of the followers and present a distributed localization law for each agent. Under the proposed position localization law, estimated positions of all followers converge globally and asymptotically to the true positions.

4.1. Unique target configuration

The uniqueness of the target configuration (the actual positions of agents) is a key property of the network that allows us to localize the network. In the sequel, under the noncollocation and non-collinearity conditions of the true positions of the agents in Assumption 2, we show that the target configuration is uniquely defined using the direction constraints, estimate of orientation of agent i, and the absolute positions of some leaders. The following lemma is similar to Trinh, et al. (2019, Lem. 1).

Lemma 6. *Consider the twin-leader–follower network with two or more leaders and locally measured directions* $\{\mathbf{b}_{ij}^i\}_{(i,j)\in\mathcal{E}}$*. Suppose that Assumptions 1–2 hold, and the orientation of agent i,* $\mathbf{R}_i \in SO(3)$*, is available to i or otherwise can be estimated, e.g. Problem 1. Then the actual position of the agent i,* $(i \ge 3)$*, i.e.,* $\mathbf{p}_i \in \mathbb{R}^3$ *is uniquely determined from its direction constraints* $\{\mathbf{b}_{ij}^i\}_{j\in\mathcal{N}_i}$ *and the positions of its neighbors* $\{\mathbf{p}_j\}_{j\in\mathcal{N}_i}$*. Furthermore,* $\mathbf{p}_i$ *is uniquely computed as*

$$\mathbf{p}_i = \Big(\sum_{j\in\mathcal{N}_i} \mathbf{P}_{\mathbf{b}_{ij}}\Big)^{-1} \sum_{j\in\mathcal{N}_i} \mathbf{P}_{\mathbf{b}_{ij}}\mathbf{p}_j, \tag{15}$$

where $\mathbf{b}_{ij} = \mathbf{R}_i\mathbf{b}_{ij}^i$*, and* $\mathbf{P}_{\mathbf{b}_{ij}} \in \mathbb{R}^{3\times 3}$ *denotes the projection matrix as defined in (1).*

4.2. Proposed position localization law

Each follower agent i holds an initial estimate of its position $\hat{\mathbf{p}}_i(0) \in \mathbb{R}^3$, and updates the estimate as follows

$$\dot{\hat{\mathbf{p}}}_i = -\hat{\mathbf{R}}_i \sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}^i} \hat{\mathbf{R}}_i^\top (\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j), \tag{16}$$

where, $k_{p_{ij}} > 0$ is a positive gain, the local projection matrix $\mathbf{P}_{\mathbf{b}_{ij}^i} = \mathbf{I}_3 - \mathbf{b}_{ij}^i(\mathbf{b}_{ij}^i)^\top = \mathbf{R}_i^\top(\mathbf{I}_3 - \mathbf{b}_{ij}\mathbf{b}_{ij}^\top)\mathbf{R}_i = \mathbf{R}_i^\top\mathbf{P}_{\mathbf{b}_{ij}}\mathbf{R}_i$, and $\hat{\mathbf{p}}_i(0)$ is initialized arbitrarily. The localization law (16) is implemented using only local direction measurements $\mathbf{b}_{ij}^i$, estimate of orientation $\hat{\mathbf{R}}_i$, and estimates of its neighbors' positions $\hat{\mathbf{p}}_j$ which are communicated by agents $j \in \mathcal{N}_i$ (in the case of leaders, $\hat{\mathbf{p}}_i = \mathbf{p}_i, \forall i \in \mathcal{V}_l$). The estimation law (16) is linear in the estimated state $\hat{\mathbf{p}}(t) := [\hat{\mathbf{p}}_1^\top(t), \ldots, \hat{\mathbf{p}}_n^\top(t)]^\top$, and so the right side is globally Lipschitz in $\hat{\mathbf{p}}(t)$.

4.3. Stability analysis

We rewrite the localization law (16) as follows

$$\dot{\hat{\mathbf{p}}}_i = \mathbf{f}_i(\hat{\mathbf{p}}, t) - \mathbf{h}_i(\hat{\mathbf{p}}, \hat{\mathbf{R}}),$$

where $\mathbf{f}_i(\hat{\mathbf{p}}, t) := -\sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}}(\hat{\mathbf{p}}_i - \mathbf{p}_j)$ and $\mathbf{h}_i(\hat{\mathbf{p}}, \hat{\mathbf{R}}) := -(\hat{\mathbf{R}}_i - \mathbf{R}_i)\sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}^i_{ij}} \mathbf{R}_i^\top(\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j) - \hat{\mathbf{R}}_i \sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}^i_{ij}}(\hat{\mathbf{R}}_i^\top - \mathbf{R}_i^\top)(\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j) - \mathbf{R}_i \sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}^i_{ij}} \mathbf{R}_i^\top(\mathbf{p}_j - \hat{\mathbf{p}}_j)$. The above dynamics can be written in a more compact form

$$\dot{\hat{\mathbf{p}}} = \mathbf{f}(\hat{\mathbf{p}}, t) + \mathbf{h}(\hat{\mathbf{p}}, t), \tag{17}$$

where the stack vectors $\mathbf{f}(\hat{\mathbf{p}}) = [\mathbf{f}_1^\top, \mathbf{f}_2^\top, \ldots, \mathbf{f}_n^\top]^\top$ and $\mathbf{h}(\hat{\mathbf{p}}, t) = [\mathbf{h}_1^\top, \mathbf{h}_2^\top, \ldots, \mathbf{h}_n^\top]^\top$. Due to the cascade structure of the system (17), we will study (17) using the stability theory for cascade systems (Angeli, 2004). Consider $\mathbf{h}(\hat{\mathbf{p}}, t)$ in (17) as an input to the following nominal system

$$\dot{\hat{\mathbf{p}}} = \mathbf{f}(\hat{\mathbf{p}}). \tag{18}$$

The boundedness of the estimates of positions is provided in the following lemma.

Lemma 7. *Under Assumptions 1–2, the cascade system (17) satisfies the ultimate boundedness property. That is, the estimates $\hat{\mathbf{p}}_i$ ($i = 3, \ldots, n$) are bounded for all $t > 0$.*

Lemma 8. *Under Assumptions 1–2, the desired equilibrium $\hat{\mathbf{p}} = \mathbf{p}$ of the nominal system (18) is globally exponentially stable (GES).*

Proof. For each follower $i \in \mathcal{V}_f$, the equilibrium of (18) satisfies $\mathbf{f}(\hat{\mathbf{p}}_i) = \mathbf{0} \Leftrightarrow \sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}}(\hat{\mathbf{p}}_i - \mathbf{p}_j) = \mathbf{0} \Leftrightarrow \left(\sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}}\right)\hat{\mathbf{p}}_i = \sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}} \mathbf{p}_j$. Since agent i is not collinear with two or more of its neighbors (Assumption 2), $\hat{\mathbf{p}}_i = \mathbf{p}_i$ is the unique solution to the equation (Lemma 6). Consequently, $\hat{\mathbf{p}} = \mathbf{p}$, is the unique equilibrium of the nominal system (18).

Consider a Lyapunov function $V_i = 1/2(\hat{\mathbf{p}}_i - \mathbf{p}_i)^2$, which is positive definite, continuously differentiable, and radially unbounded. It can be shown that the derivative of V_i along the trajectory of (18) is given as $\dot{V}_i(t) = -(\hat{\mathbf{p}}_i - \mathbf{p}_i)^\top\left(\sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}}\right)(\hat{\mathbf{p}}_i - \mathbf{p}_i)$, which is negative definite due to the positive definiteness of $\left(\sum_{j\in\mathcal{N}_i} k_{p_{ij}} \mathbf{P}_{\mathbf{b}_{ij}}\right)$ (Lemma 6). This completes the proof. □

Theorem 3. *Under Assumptions 1–2, the cascade system (17) is input-to-state stable with respect to the input $\mathbf{h}(\hat{\mathbf{p}}, \hat{\mathbf{R}})$. Further, $\hat{\mathbf{p}}(t) \to \mathbf{p}$ almost globally and asymptotically as $t \to \infty$.*

Proof. We provide a proof by using mathematical induction. The almost global asymptotic convergence of the localized position of the first follower follows directly since the desired equilibrium, $\hat{\mathbf{p}}_3 = \mathbf{p}_3$, of the nominal system, $\dot{\hat{\mathbf{p}}}_3 = \mathbf{f}_3(\hat{\mathbf{p}}_3, t)$, is GES (Lemma 8), and the input is bounded and $\mathbf{h}_3(t) \to 0$ asymptotically due to $\hat{\mathbf{R}}_i \to \mathbf{R}_i$ almost globally as $t \to \infty$ and $\hat{\mathbf{p}}_j = \mathbf{p}_j, \forall j \in \mathcal{N}_3$. It can be shown similarly for all other followers using the facts that the convergence of position estimate of an follower is not influenced by the latter agents in the network and the orientations and positions of earlier agents converge to the actual poses asymptotically. This completes the proof. □

Remark 1. The position localization (16) runs in parallel with the aforementioned orientation estimation scheme (4). Note also that the underlying graph of a twin-leader–follower network is bearing rigid. Thus, given estimates of agent orientations $\hat{\mathbf{R}}_i(t)$, the position computations in the network via (16) can also be done in a bidirectional way. Then, we obtain Corollary 2 whose proof is similar to Proof of Li, Luo, and Zhao (2020, Thm. 2). We showed above that the position estimation via (16) using unidirectional communications is also almost globally convergent using the input-to-state stability theory.

Corollary 2. *Suppose that Assumptions 1–2 hold and the sum in (16) is taken over all j to which agent i measures directions $\mathbf{b}^i_{ij}$. Then, under the estimation law (16), $\hat{\mathbf{p}}(t) \to \mathbf{p}$ almost globally and asymptotically as $t \to \infty$.*

5. Conclusion

In this paper, a network pose localization scheme was proposed for *twin-leader–follower* networks by using direction measurements in $\mathbb{R}^3$. In particular, an orientation localization law in $SO(3)$ and a position localization protocol were presented. We showed that the actual orientations and positions of all follower agents can be estimated almost globally and asymptotically. An extension of this work to systems with more general graph topologies is left as future work.

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