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Generalized weak rigidity: Theory, and local and global convergence of formations[☆]

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ABSTRACT

This paper proposes a generalized weak rigidity theory, and aims to apply the theory to formation control problems with a gradient descent flow law. The generalized weak rigidity theory is utilized in order to characterize desired rigid formations by a general set of pure inter-agent distances and subtended angles, where the rigid formation shape with distances and subtended angles is determined up to translations and rotations (if the formation shape is composed of only subtended angles, then it is determined up to translations, rotations and, additionally, scaling factors). As the first result of its applications, this paper provides analysis of local exponential stability for a formation control system with pure distance/angle or only angle constraints in 2- and 3-dimensional spaces. Then, as the second result, it is shown that if there are three agents in 2-dimensional space then almost global exponential stability is ensured for a formation control system with pure distance/angle or only angle constraints.

1. Introduction

Based on the rigidity theories, distributed formation control has been investigated under the networked multi-agent systems [1–4]. In formation control problems, the rigidity theories have been key concepts to characterize a rigid formation shape[1] with a specific set of constraints, such as distances, bearings, subtended angles, etc. The rigidity theories can briefly be classified according to types of constraints; for example, distance based rigidity theory, bearing based rigidity theory, angle based rigidity theory and mixed rigidity theory.

In particular, based on use of the distance based rigidity (distance rigidity) theory [5–8], formation control problems have been extensively studied [3,9–12], where a rigid formation is characterized by constraints of inter-agent distances. In formation control with the distance rigidity theory, each agent is required to sense relative positions of its neighbors. In terms of the bearing based rigidity (bearing rigidity or parallel rigidity) theory [13–16], inter-agent bearings are used to achieve a unique formation shape (up to translations and scaling factors) with which formation control problems have been also studied [15–17]. This approach makes use of measurement of relative bearings or positions of its neighbors in formation control. In recent years, formation control problems based on the angle based rigidity theory and mixed rigidity theory have attracted much research interest [2,18–25].[2]

This paper particularly focuses on formation control based on the mixed rigidity theory with distances and subtended angles, where the rigidity theory with distances and subtended angles is called weak rigidity theory [19–22]. In addition, formation control with only subtended angles is also of interest to this paper. Although there have been several studies on formation control with angle information over undirected sensing networks [26–29], such studies are not completed yet. For example, the works [26–28] propose 3-agent formation control laws to achieve local exponential convergence or global asymptotic convergence of 3-agent formations in 2-dimensional space, and the work [29] only considers local exponential convergence of multi-agent formations in 2-dimensional space. Compared with the existing formation control problems involving only subtended angles, we will show that our proposed control law can guarantee local exponential convergence of multi-agent formations in 3-dimensional space as well as 2-dimensional space, and global exponential convergence of 3-agent formations in 2-dimensional space. The main motivation on studying such formation control is that distance constraints (edges) in characterizing rigid formations (graphs) can be removed, which leads to the reduction of

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1 In this paper, a formation shape is said to be rigid if smooth motions of all agents are those corresponding to trivial motions, such as translation, rotation and scaling of the entire formation, without deformation of the formation shape.

2 The concept of the stiffness introduced in [18,24,25] could be regarded as the concept of the rigidity.

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the number of agents that control scaling factors of a formation in formation control. This motivation is explained in detail in Remark 1 in Section 2.

In fact, although the weak rigidity theories in [19–22] are conceptually similar in the sense that subtended angle information is used, the weak rigidity theories have been interpreted in a different way. To distinguish the existing relevant works, we call the theories of Park et al. (2017) [19], Jing et al. (2018) [20] and Kwon et al. (2018) [22] *basic weak rigidity theory*, *type-1 weak rigidity theory* and *type-2 weak rigidity theory*, respectively, in this paper. In the work based on the basic weak rigidity theory [19], the authors introduced the weak rigidity theory for the first time, where the theory is studied with some special cases in the 2-dimensional space. In accordance with the definition of the basic weak rigidity theory, a rigid formation has to be composed of triangular formations, and each triangular formation should have two adjacency distance constraints to define a subtended angle constraint. For example, as shown in Fig. 1(a), two distance constraints for a subtended angle constraint should be defined for the triangular formation. Based on the type-1 weak rigidity theory [20], inner products of inter-agent relative positions can be regarded as angle constraints to characterize rigid formations; however, such an inner product cannot be regarded as pure angle information. We would like to remark that the inner products of inter-agent relative positions are distinct from the inner products of inter-agent relative bearings, i.e., cosines of subtended angles among agents. The inner product of inter-agent relative positions used in the type-1 weak rigidity theory includes distance and angle information simultaneously, which implies that it could include redundant information when characterizing rigid formations. For example, considering two inner products $z_{21}^\top z_{31}$ and $z_{13}^\top z_{23}$, where z_{ij} denotes a relative position from agent j to agent i, we can observe that the Euclidean norm of z_{13} is redundantly involved. Moreover, the type-1 weak rigidity theory cannot consider rigid formations with only subtended angle information. In recent years, the type-2 weak rigidity theory [21,22] has been introduced, where the concept of the type-2 weak rigidity theory is the extended concept from the basic weak rigidity theory but distinguished from the type-1 weak rigidity theory by types of constraints. Compared with the type-1 weak rigidity theory, the type-2 weak rigidity theory involves pure distance/angle or only angle constraints without any redundant information; for example, see Fig. 1(b). In particular, based on the type-2 weak rigidity theory, one can achieve a rigid formation with only subtended angle constraints as shown in Fig. 1(c) whereas one cannot achieve it based on the type-1 weak rigidity theory. The comparison between the type-1 and type-2 weak rigidity theories is again highlighted in Remark 2 in Section 3.

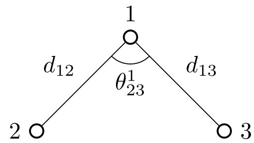
Based on the type-1 weak rigidity theory, the studies on multi-agent formation control in the d-dimensional space are almost completed in [20]. On the other hand, there are still many tasks that need to be studied in the case of the type-2 weak rigidity theory in d-dimensional space. In this sense, this paper aims to explore the type-2 weak rigidity theory and, further, to apply the theory to formation control. In this paper, to differentiate between the weak rigidity theories, the extended concept from the type-2 weak rigidity theory is named *generalized weak rigidity*. Consequently, the main contributions of this paper are summarized as follows. First, we introduce the concepts of *generalized weak rigidity* and *generalized infinitesimal weak rigidity* in 2- and 3-dimensional spaces. These concepts are used to examine whether or not a given formation with pure distance/angle or only angle constraints is rigid or globally rigid. We then show that both concepts are generic properties. Moreover, it is shown that the generalized weak rigidity theory is a weaker condition than the conventional distance rigidity theory. Second, we apply the generalized weak rigidity theory to formation control with a gradient descent flow law. Based on the generalized weak rigidity theory, we provide analysis of local exponential stability for a n-agent formation control system in 2- and 3-dimensional spaces, and further analysis of almost global exponential stability for a 3-agent formation control system in 2-dimensional space.

The rest of this paper is organized as follows. Preliminaries, notations and motivation are briefly given in Section 2. Then, Section 3 presents the generalized weak rigidity theory. Based on the rigidity theory, Sections 4 and 5 discuss analysis of local convergence and almost global convergence of formations, respectively. Finally, Section 6 provides conclusion and summary.

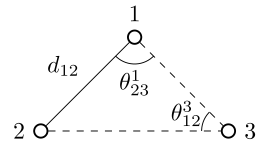
2. Preliminary

Let $\|\cdot\|$ and $|\mathcal{S}|$ denote the Euclidean norm of a vector and cardinality of a set $\mathcal{S}$, respectively. The symbols $\mathrm{Null}(\cdot)$ and $\mathrm{rank}(\cdot)$ denote the null space and rank of a matrix, respectively. The symbol $I_N \in \mathbb{R}^{N\times N}$ denotes the identity matrix, and the symbol $\mathbb{1}_n \in \mathbb{R}^n$ denotes a vector whose all entries are 1 as $\mathbb{1}_n = [1,\ldots,1]^\top$. We define an undirected graph $\mathcal{G}$ as $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,2,\ldots,n\}$ denotes a vertex set and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ denotes an edge set with $m = |\mathcal{E}|$. Since an undirected graph is considered, it is assumed that $(i,j) = (j,i)$ for all $i,j \in \mathcal{V}$. An angle set $\mathcal{A} \subseteq \mathcal{V}\times\mathcal{V}\times\mathcal{V}$ is defined as $\mathcal{A} = \{(k,i,j) \mid \theta_{ij}^k \text{ is assigned to } i,j,k \in \mathcal{V}, \theta_{ij}^k \in [0,\pi]\}$ with $w = |\mathcal{A}|$, where θ_{ij}^k denotes an angle subtended by the adjacent edges (i,k) and (j,k), where the adjacent edges (i,k) and (j,k) do not necessarily belong to $\mathcal{G}$. Angles used in this paper have no directions and signs. For a position vector $p_i \in \mathbb{R}^d$, we define a configuration p of $\mathcal{G}$ as $p = [p_1^\top,\ldots,p_n^\top]^\top \in \mathbb{R}^{dn}$ and define a framework as $(\mathcal{G},\mathcal{A},p)$ in $\mathbb{R}^d$. A formation is regarded as a framework in this paper. We define a relative position vector as $z_{ij} = p_i - p_j$ for a framework $(\mathcal{G},\mathcal{A},p)$, $(i,j) \in \mathcal{E}$ and $i \neq j$. We set the order of the associated relative position vectors z_{ij} as $z_{g_{ij}} = z_{ij}, g \in \{1,\ldots,m\}$. Similarly, for $(k,i,j) \in \mathcal{A}$ and $h \in \{1,\ldots,w\}$, a cosine $A_{h_{kij}}$ is defined as $A_{h_{kij}} = \cos\theta_{ij}^k$. It is remarkable that $A_{h_{kij}}$ is equivalently represented as $A_{h_{kij}} = \cos\theta_{ij}^k = \frac{z_{ki}^\top z_{kj}}{\|z_{ki}\|\|z_{kj}\|} = \frac{\|z_{ki}\|^2+\|z_{kj}\|^2-\|z_{ij}\|^2}{2\|z_{ki}\|\|z_{kj}\|}$. We occasionally make use of z_g and A_h for notational convenience instead of $z_{g_{ij}}$ and $A_{h_{kij}}$, respectively, if no confusion is expected. Note that, in this paper, we focus on problems only in 2- and 3-dimensional spaces, i.e., $d = 2, 3$.

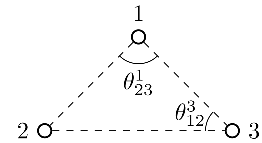
Remark 1. The advantages of formation control studied in this paper are mainly threefold. First, the proposed formation control protocol with pure distance/angle constraints is convenient to control scalings of formations compared with the formation control system composed of only distance constraints; for example, when we want to control a scaling of the formation illustrated in Fig. 2(b), we only need to control the distance constraint between agents 1 and 2 while all distance constraints of the formation illustrated in Fig. 2(a) have to be controlled. This is due to the fact that pure angle constraints are invariant to trivial motions corresponding to translations, rotations and scalings of an entire formation while distance constraints are invariant to only a subset of the motions, i.e., translations and rotations. Second, the proposed control system is a distributed multi-agent system, that is, each agent only needs to measure relative positions of its neighbor agents with respect to its local coordinate system. Third, orientations of agents do not need to be aligned and each agent does not require any orientation information in formation control. These advantages can be found in Sections 4 and 5.



(a) Triangular formation characterized by two distance constraints and one angle constraint subtend by the two distance constraints.

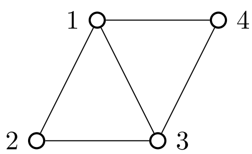


(b) Triangular formation characterized by one distance constraint and two subtended angle constraints.

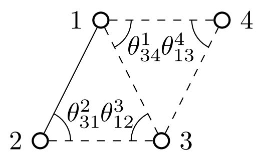


(c) Triangular formation characterized by two subtended angle constraints.

Fig. 1. Triangular formations with different constraints. The symbol d_{ij} denotes a distance constraint between vertices i and j, and the symbol θ^k_{ij} denotes an angle constraint subtended by edges (i,k) and (j,k). The dashed lines indicate virtual edges which are not distance constraints.



(a) Rigid formation with pure distance constraints



(b) Rigid formation with pure distance and angle constraints

Fig. 2. Examples of rigid formations in $\mathbb{R}^2$, where the solid lines denote distance constraints, and the dashed lines denote virtual edges which are not distance constraints. Angle constraints are denoted by $\theta^k_{ij}, (k,i,j) \in \mathcal{A}$.

3. Generalized weak rigidity

In this section, we introduce a generalized weak rigidity theory in $\mathbb{R}^d$. The basic concept on the theory is related to how to examine whether or not a rigid formation shape can be determined up to a translation and a rotation (and additionally, for specific cases, a scaling factor) by given relative distance and subtended angle constraints.

3.1. Generalized weak rigidity (GWR)

In order to define the concept of the generalized weak rigidity, we make use of the following definition used in the distance rigidity theory. It is well known that two frameworks $(\mathcal{G}, \mathcal{A}, p)$ and $(\mathcal{G}, \mathcal{A}, q)$ are said to be *congruent* if $\|p_i - p_j\| = \|q_i - q_j\|$ for all $i, j \in \mathcal{V}$. We now define the fundamental concepts on the generalized weak rigidity.

Definition 1 (*Strong Equivalency*)**.** Two frameworks $(\mathcal{G}, \mathcal{A}, p)$ and $(\mathcal{G}, \mathcal{A}, q)$ are said to be *strongly equivalent* if the following two conditions hold

- $\|p_i - p_j\| = \|q_i - q_j\|, \forall (i,j) \in \mathcal{E}$,
- $\cos\left(\theta^k_{ij}\right)_{\in(\mathcal{G},\mathcal{A},p)} = \cos\left(\theta^k_{ij}\right)_{\in(\mathcal{G},\mathcal{A},q)}, \forall (k,i,j) \in \mathcal{A}$,

where $\left(\theta^k_{ij}\right)_{\in(\mathcal{G},\mathcal{A},p)}$ and $\left(\theta^k_{ij}\right)_{\in(\mathcal{G},\mathcal{A},q)}$ denote the angles belonging to $(\mathcal{G}, \mathcal{A}, p)$ and $(\mathcal{G}, \mathcal{A}, q)$, respectively.

Definition 2 (*Angle Equivalency*)**.** Two frameworks $(\mathcal{G}, \mathcal{A}, p)$ and $(\mathcal{G}, \mathcal{A}, q)$ with $\mathcal{E} = \emptyset$ are said to be *angle equivalent* if $\cos\left(\theta^k_{ij}\right)_{\in(\mathcal{G},\mathcal{A},p)} = \cos\left(\theta^k_{ij}\right)_{\in(\mathcal{G},\mathcal{A},q)}, \forall (k,i,j) \in \mathcal{A}$.

In this paper, $\mathcal{E} \neq \emptyset$ means that there exists at least one distance constraint; on the other hand, $\mathcal{E} = \emptyset$ means that any distance constraint does not exist.

Definition 3 (*Proportional Congruency*)**.** Two frameworks $(\mathcal{G}, \mathcal{A}, p)$ and $(\mathcal{G}, \mathcal{A}, q)$ with $\mathcal{E} = \emptyset$ are said to be *proportionally congruent* if $\|p_i - p_j\| = C\|q_i - q_j\|, \forall i, j \in \mathcal{V}$, where C denotes a positive proportional constant.

Definition 4 (*Generalized Weak Rigidity (GWR)*)**.** A framework $(\mathcal{G}, \mathcal{A}, p)$ is *generalized weakly rigid (GWR)* in $\mathbb{R}^d$ if there exists a neighborhood $\mathcal{B}_p \subseteq \mathbb{R}^{dn}$ of p such that each framework $(\mathcal{G}, \mathcal{A}, q)$, $q \in \mathcal{B}_p$, strongly equivalent to $(\mathcal{G}, \mathcal{A}, p)$ is congruent to $(\mathcal{G}, \mathcal{A}, p)$. Moreover, a framework $(\mathcal{G}, \mathcal{A}, p)$ with $\mathcal{E} = \emptyset$ is also *generalized weakly rigid (GWR)* in $\mathbb{R}^d$ if there exists a neighborhood $\mathcal{B}_p \subseteq \mathbb{R}^{dn}$ of p such that each framework $(\mathcal{G}, \mathcal{A}, q)$, $q \in \mathcal{B}_p$, angle equivalent to $(\mathcal{G}, \mathcal{A}, p)$ is proportionally congruent to $(\mathcal{G}, \mathcal{A}, p)$.

Definition 5 (*Global GWR*)**.** A framework $(\mathcal{G}, \mathcal{A}, p)$ is *globally GWR* in $\mathbb{R}^d$ if any framework $(\mathcal{G}, \mathcal{A}, q)$ strongly equivalent to $(\mathcal{G}, \mathcal{A}, p)$ is congruent to $(\mathcal{G}, \mathcal{A}, p)$. Moreover, a framework $(\mathcal{G}, \mathcal{A}, p)$ with $\mathcal{E} = \emptyset$ is also *globally GWR* in $\mathbb{R}^d$ if any framework $(\mathcal{G}, \mathcal{A}, q)$ angle equivalent to $(\mathcal{G}, \mathcal{A}, p)$ is proportionally congruent to $(\mathcal{G}, \mathcal{A}, p)$.

If a framework is GWR (resp. globally GWR), then the framework shape is (resp. globally) rigid and not deformable up to translations and rotations of a given framework for $\mathcal{E} \neq \emptyset$ or up to translations, rotations and scalings for $\mathcal{E} = \emptyset$. Fig. 3 shows several examples of GWR and non-GWR formations in $\mathbb{R}^2$. The formations represented in Figs. 3(a), 3(b) and 3(d) are GWR since they cannot be deformed (in Fig. 3(b), a deformed formation by scaling is also regarded as a GWR formation). In particular, the formation in Fig. 3(a) is globally GWR, and thus its shape is globally determined up to translations and rotations. The formation in Fig. 3(b) is not globally GWR but GWR since it is rigid but the agent 4 (or agent 2) can be flipped over edge $(1,3)$ while all angle constraints maintain the values. Similarly, the formation in Fig. 3(d) is not globally GWR but GWR. On the other hand, the formation represented in Fig. 3(c) is neither GWR nor globally GWR since it can be deformed by a smooth motion on a circle containing vertices 1, 3 and 4.

3.2. Generalized infinitesimal weak rigidity (GIWR)

We now introduce the concept of the *generalized infinitesimal weak rigidity* which plays an important role in formation control studied in this paper. To define the concept, we first introduce a *weak rigidity matrix* with which we can check whether or not a formation is rigid by an algebraic manner, i.e., rank condition of the weak rigidity matrix.

We define the following *weak rigidity function* $F_W : \chi \to \mathbb{R}^{m+w}$ for $\chi \subset \mathbb{R}^{dn}$, where χ is well defined not to make a denominator in $A_h, h \in \{1, \ldots, w\}$ zero, which describes constraints of edge lengths and angles in a framework:

$$F_W(p) = \left[\|z_1\|^2, \ldots, \|z_m\|^2, A_1, \ldots, A_w\right]^\top \in \mathbb{R}^{m+w}. \tag{1}$$

We then define the following *weak rigidity matrix* as the Jacobian of the weak rigidity function:

$$R_W(p) = \frac{\partial F_W(p)}{\partial p} = \begin{bmatrix} \frac{\partial \mathbf{D}}{\partial p} \\ \frac{\partial \mathbf{A}}{\partial p} \end{bmatrix} \in \mathbb{R}^{(m+w)\times dn}, \tag{2}$$

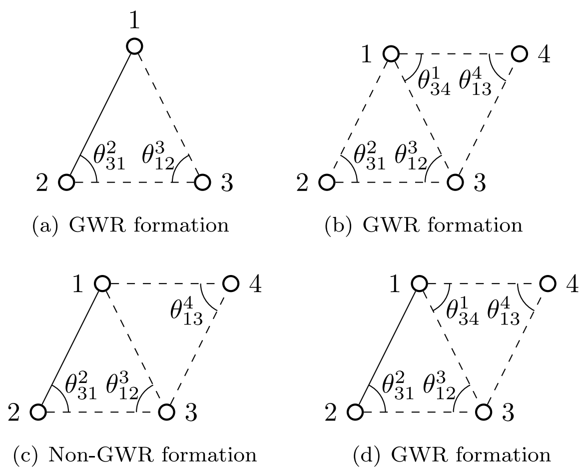


Fig. 3. Examples of GWR and non-GWR formations in $\mathbb{R}^2$. The solid lines denote distance constraints belonging to $\mathcal{E}$, but the dashed lines which do not belong to $\mathcal{E}$ are not distance constraints.

where $\mathbf{D} = \left[\|z_1\|^2, \|z_2\|^2, \ldots, \|z_m\|^2\right]^\top \in \mathbb{R}^m$ and $\mathbf{A} = [A_1, A_2, \ldots, A_w]^\top \in \mathbb{R}^w$. Next, consider the constraints

$$\|p_i - p_j\|^2 = constant, \ \forall (i,j) \in \mathcal{E}, \tag{3}$$

$$\cos\theta_{ij}^k = constant, \ \forall (k,i,j) \in \mathcal{A}. \tag{4}$$

Then, the time derivative of (3) is given by

$$2\left(p_i - p_j\right)^\top \left(v_i - v_j\right) = 0, \ \forall (i,j) \in \mathcal{E}, \tag{5}$$

and the time derivative of (4) is given as

$$\begin{aligned}&\left(v_k - v_i\right)^\top P_{z_{ki}}^\top \frac{z_{kj}}{\|z_{kj}\|} + \frac{z_{ki}^\top}{\|z_{ki}\|} P_{z_{kj}} \left(v_k - v_j\right) \\ &= 0, \ \forall (k,i,j) \in \mathcal{A},\end{aligned} \tag{6}$$

where v_i is an infinitesimal motion of vertex i, and $P_{z_{ki}} = \frac{1}{\|z_{ki}\|}\left[I_d - \frac{z_{ki}z_{ki}^\top}{\|z_{ki}\|^2}\right]$ and $P_{z_{kj}} = \frac{1}{\|z_{kj}\|}\left[I_d - \frac{z_{kj}z_{kj}^\top}{\|z_{kj}\|^2}\right]$. For both cases $\mathcal{E} \neq \emptyset$ and $\mathcal{E} = \emptyset$, Eqs. (5) and (6) can be written in a matrix form as $\dot{F}_W = \frac{\partial F_W(p)}{\partial p}\dot{p} = R_W(p)\dot{p} = 0$. We here denote an infinitesimal motion of $(\mathcal{G}, \mathcal{A}, p)$ by δp if $R_W(p)\delta p = 0$. The infinitesimal motions include rigid-body translations and rotations when $\mathcal{E} \neq \emptyset$. If $\mathcal{E} = \emptyset$ then the infinitesimal motions additionally include scalings, that is, the motions include rigid-body translations, rotations and scalings. We finally have the concept of the generalized infinitesimal weak rigidity with the following definition of the trivial infinitesimal motion.

Definition 6 (*Trivial Infinitesimal Motion [21]*). An infinitesimal motion of a framework $(\mathcal{G}, \mathcal{A}, p)$ is called *trivial* if it corresponds to a rigid-body translation or a rigid-body rotation (or additionally, when $\mathcal{E} = \emptyset$, a scaling factor) of the entire framework.

Definition 7 (*Generalized Infinitesimal Weak Rigidity (GIWR)*). A framework $(\mathcal{G}, \mathcal{A}, p)$ is *generalized infinitesimally weakly rigid (GIWR)* in $\mathbb{R}^d$ if all of its infinitesimal motions are trivial.

We next explore the properties of GIWR formations. For $d = 2$ case, it is already shown that the GIWR can be checked by the rank condition of R_W as in [21]. Therefore, we explore the properties only for $d = 3$ case. We first express the trivial infinitesimal motions in mathematical forms. For $d = 3$ case, we define the rotational matrix J_i, $\forall i \in \{1,2,3\}$ as

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

Note it always holds that $x^\top J_i x = 0, \forall i \in \{1,2,3\}$ for any vector $x \in \mathbb{R}^3$. Referring to Lemma 1 in [30], we have that the vectors in the following set, L_R, are linearly independent.

$$L_R = \{\mathbb{1}_n \otimes I_3, (I_n \otimes J_1)p, (I_n \otimes J_2)p, (I_n \otimes J_3)p\}, \tag{8}$$

where $(\mathbb{1}_n \otimes I_3)$ and $(I_n \otimes J_i)p$, $i \in \{1,2,3\}$ correspond to a rigid-body translation and a rigid-body rotation of an entire framework, respectively. We define a set L_N for a rigid-body translation, a rigid-body rotation and a scaling of a framework in $\mathbb{R}^3$ as

$$L_N = \{\mathbb{1}_n \otimes I_3, (I_n \otimes J_1)p, (I_n \otimes J_2)p, (I_n \otimes J_3)p, p\}. \tag{9}$$

The sets L_R and L_N can be regarded as the bases for d-dimensional rigid transformations and similarity transformations of a formation, respectively. Moreover, it is obvious that any linear combination of the vectors in L_R cannot be equal to span$\{p\}$ since a framework induced from span$\{L_R\}$ is embedded in the 3-dimensional group of rigid transformations, i.e., Special Euclidean group $SE(3)$, which means that rigid transformations span$\{L_R\}$ cannot be equal to nonrigid transformations span$\{p\}$. Hence, the vectors in the set L_N are also linearly independent.

We state some notations to prove Lemmas 2 and 3 presented in what follows. We first define a graph $\mathcal{G}'$ as $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{A}')$ induced from $\mathcal{G}$ in such a way that:

- $\mathcal{V}' = \mathcal{V}$,
- $\mathcal{E}' = \{(i,j), (i,k), (j,k) \mid (i,j) \in \mathcal{E} \vee (k,i,j) \in \mathcal{A}\}$,
- $\mathcal{A}' = \mathcal{A}$.

For any edge $(i,j) \in \mathcal{E}'$, we consider a new associated relative position vector z'_{ij}, and set the order of the new relative position vector as follows:

$$z'_s = z'_{ij}, \forall s \in \{1, \ldots, \eta\}, \eta \geq m,$$

where $z'_{ij} = p_i - p_j$ for all $(i,j) \in \mathcal{E}'$, and $\eta = |\mathcal{E}'|$. The anew defined relative position vector satisfies the following condition

$$z'_u = z_u, \forall u \in \{1, \ldots, m\}.$$

We denote a new associated column vector composed of relative position vectors as $z' = \left[z'^\top_1, z'^\top_2, \ldots, z'^\top_\eta\right]^\top \in \mathbb{R}^{3\eta}$. The oriented incidence matrix $H' \in \mathbb{R}^{\eta \times n}$ of the induced graph $\mathcal{G}'$ is the $\{0, \pm 1\}$-matrix with rows indexed by edges and columns indexed by vertices as follows:

$$[H']_{si} = \begin{cases} 1 & \text{if the } s\text{th edge sinks at vertex } i \\ -1 & \text{if the } s\text{th edge leaves vertex } i \\ 0 & \text{otherwise,} \end{cases}$$

where $[H']_{si}$ is an element at row s and column i of the matrix H'. Note that z' satisfies $z' = \bar{H}'p$ where $\bar{H}' = H' \otimes I_d$. We are now ready to define the following properties.

Lemma 1 (*[21, Lemma 3.3]*). *Let J_0 denote a rotational matrix defined as $J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in $\mathbb{R}^2$. For $d = 2$ case, it is satisfied that* span$\{\mathbb{1} \otimes I_2, (I_n \otimes J_0)p\} \subseteq Null(R_W(p))$ *and* rank$(R_W(p)) \leq 2n - 3$ *if $\mathcal{E} \neq \emptyset$. In addition, for $d = 2$ case, it is satisfied that* span$\{\mathbb{1} \otimes I_2, (I_n \otimes J_0)p, p\} \subseteq Null(R_W(p))$ *and* rank$(R_W(p)) \leq 2n - 4$ *if $\mathcal{E} = \emptyset$.*

Lemma 2. *For $d = 3$ case, it is satisfied that, when $\mathcal{E} \neq \emptyset$ and $\mathcal{E} = \emptyset$,* span$(L_R) \subseteq$ Null$(R_W(p))$ *and* span$(L_N) \subseteq$ Null$(R_W(p))$*, respectively.*

Proof. This property is proved by a similar approach to Lemma 1. When $\mathcal{E} \neq \emptyset$, Eq. (2) can be written as

$$R_W(p) = \frac{\partial F_W(p)}{\partial p} = \begin{bmatrix} \frac{\partial \mathbf{D}}{\partial z'}\frac{\partial z'}{\partial p} \\ \frac{\partial \mathbf{A}}{\partial z'}\frac{\partial z'}{\partial p} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{D}}{\partial z'}\bar{H}' \\ \frac{\partial \mathbf{A}}{\partial z'}\bar{H}' \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{D}}{\partial z'} \\ \frac{\partial \mathbf{A}}{\partial z'} \end{bmatrix}\bar{H}'. \tag{10}$$

Then, it is obvious that $\text{span}\{\mathbb{1}_n \otimes I_3\} \subseteq \text{Null}(\bar{H}') \subseteq \text{Null}(R_W(p))$ since $\text{span}\{\mathbb{1}_n\} \subseteq \text{Null}(H')$. We next check whether $R_W(p)(I_n \otimes J_i)p = 0$ or not. $\bar{H}'(I_n \otimes J_i)p, \forall i \in \{1,2,3\}$ can be of such form

$$\begin{aligned}\bar{H}'(I_n \otimes J_i)p &= (H' \otimes I_3)(I_n \otimes J_i)p = (H' \otimes J_i)p \\ &= (I_\eta H' \otimes J_i I_3)p = (I_\eta \otimes J_i)(H' \otimes I_3)p \\ &= (I_\eta \otimes J_i)z' = \begin{bmatrix} J_i z'_1 \\ \vdots \\ J_i z'_\eta \end{bmatrix}. \end{aligned} \tag{11}$$

From the viewpoint of $A_h = \frac{\|z_{ki}\|^2 + \|z_{kj}\|^2 - \|z_{ij}\|^2}{2\|z_{ki}\|\|z_{kj}\|}, (k,i,j) \in \mathcal{A}$, if A_h consists of z'_a, z'_b and z'_c for $a \neq b \neq c$ and $a,b,c \in \{1,\ldots,\eta\}$ then almost all elements of $\frac{\partial A_h}{\partial z'}$ are zero except for $\frac{\partial A_h}{\partial z'_a}$, $\frac{\partial A_h}{\partial z'_b}$ and $\frac{\partial A_h}{\partial z'_c}$. With reference to the form of $\frac{\partial A_h}{\partial z'}$ as presented in Lemma 3.1 in [21], we have

$$\begin{aligned}\frac{\partial A_h}{\partial z'}\bar{H}'(I_n \otimes J_i)p &= \frac{\partial A_h}{\partial z'}\begin{bmatrix} J_i z'_1 \\ \vdots \\ J_i z'_\eta \end{bmatrix} \\ &= \frac{\partial A_h}{\partial z'_a}J_i z'_a + \frac{\partial A_h}{\partial z'_b}J_i z'_b + \frac{\partial A_h}{\partial z'_c}J_i z'_c \\ &= 0, \end{aligned} \tag{12}$$

where ${z'_a}^\top J_i z'_a = 0$, ${z'_b}^\top J_i z'_b = 0$ and ${z'_c}^\top J_i z'_c = 0$ for all $i \in \{1,2,3\}$. Thus, $\frac{\partial \mathbf{A}}{\partial z'}\bar{H}'(I_n \otimes J_i)p = 0$. We also have

$$\begin{aligned}\frac{\partial \mathbf{D}}{\partial z'}\bar{H}'(I_n \otimes J_i)p &= \frac{\partial \mathbf{D}}{\partial z'}\begin{bmatrix} J_i z'_1 \\ \vdots \\ J_i z'_\eta \end{bmatrix} \\ &= \begin{bmatrix} 2D^\top & 0_{m,(3\eta-3m)} \end{bmatrix}\begin{bmatrix} J_i z'_1 \\ \vdots \\ J_i z'_\eta \end{bmatrix} \\ &= 0, \end{aligned} \tag{13}$$

where $D = \text{diag}(z'_1,\ldots,z'_m) \in \mathbb{R}^{3m \times m}$, and $0_{m,(3\eta-3m)}$ is a $m \times (3\eta - 3m)$ zero matrix. Using the above results, we have

$$R_W(p)(I_n \otimes J_i)p = 0, \forall i \in \{1,2,3\}, \tag{14}$$

which implies that, when $\mathcal{E} \neq \emptyset$, $\text{span}\{(I_n \otimes J_i)p\} \subseteq \text{Null}(R_W(p))$, $\forall i \in \{1,2,3\}$.

If $\mathcal{E} = \emptyset$, then $R_W(p)$ is of the form

$$R_W(p) = \frac{\partial F_W(p)}{\partial p} = \frac{\partial \mathbf{A}}{\partial z'}\bar{H}'. \tag{15}$$

Then, $R_W(p)p = \frac{\partial \mathbf{A}}{\partial z'}\bar{H}'p = \frac{\partial \mathbf{A}}{\partial z'}z'$. With reference to Lemma 3.1 in [21], the elements of $\frac{\partial A_h}{\partial z'}$ are zero except for $\frac{\partial A_h}{\partial z'_a}$, $\frac{\partial A_h}{\partial z'_b}$ and $\frac{\partial A_h}{\partial z'_c}$, and we have the following result:

$$\begin{aligned}\frac{\partial A_h}{\partial z'}z' &= \frac{\partial A_h}{\partial z'}\begin{bmatrix} z'_1 \\ \vdots \\ z'_\eta \end{bmatrix} \\ &= \frac{\partial A_h}{\partial z'_a}z'_a + \frac{\partial A_h}{\partial z'_b}z'_b + \frac{\partial A_h}{\partial z'_c}z'_c \\ &= \frac{\|z'_a\|^2 - \|z'_b\|^2 + \|z'_c\|^2}{2\|z'_a\|\|z'_b\|} + \frac{-\|z'_a\|^2 + \|z'_b\|^2 + \|z'_c\|^2}{2\|z'_a\|\|z'_b\|} \\ &\quad + \frac{-2\|z'_c\|^2}{2\|z'_a\|\|z'_b\|} \\ &= 0. \end{aligned} \tag{16}$$

Thus, we have $R_W(p)p = 0$, which implies that $\text{span}\{p\} \subseteq \text{Null}(R_W(p))$. It also holds that, when $\mathcal{E} = \emptyset$, $\text{span}\{\mathbb{1}_n \otimes I_3\} \subseteq \text{Null}(R_W(p))$ and $\text{span}\{(I_n \otimes J_i)p\} \subseteq \text{Null}(R_W(p)), \forall i \in \{1,2,3\}$ in the same way as the case of $\mathcal{E} \neq \emptyset$. Consequently, the statement is proved. □

Lemma 3. *If $\mathcal{E} \neq \emptyset$, then $\text{rank}(R_W(p)) \leq dn - d(d+1)/2$ for a framework $(\mathcal{G}, \mathcal{A}, p)$ in $\mathbb{R}^d$. On the other hand, if $\mathcal{E} = \emptyset$, then $\text{rank}(R_W(p)) \leq dn - (d^2+d+2)/2$ for a framework $(\mathcal{G}, \mathcal{A}, p)$ in $\mathbb{R}^d$.*

Proof. For $d = 2$ case, it holds that $\text{rank}(R_W(p)) \leq dn - d(d+1)/2$ and $\text{rank}(R_W(p)) \leq dn - (d^2+d+2)/2$ when $\mathcal{E} \neq \emptyset$ and $\mathcal{E} = \emptyset$, respectively, from Lemma 1.

For $d = 3$ case, from Lemma 2, we have $\text{span}(L_R) \subseteq \text{Null}(R_W(p))$ when $\mathcal{E} \neq \emptyset$, which implies that $\text{rank}(R_W(p)) \leq dn - d(d+1)/2$ since the vectors in L_R are linearly independent. Similarly, when $\mathcal{E} = \emptyset$, we have $\text{span}(L_N) \subseteq \text{Null}(R_W(p))$, which implies that $\text{rank}(R_W(p)) \leq dn - (d^2+d+2)/2$ since the vectors in L_N are linearly independent. □

The following result shows the necessary and sufficient condition for the GIWR.

Theorem 1. *A framework $(\mathcal{G}, \mathcal{A}, p)$ with $n \geq 3$ and $\mathcal{E} \neq \emptyset$ is GIWR in $\mathbb{R}^d$ if and only if the weak rigidity matrix $R_W(p)$ has rank $dn - d(d+1)/2$. In addition, a framework $(\mathcal{G}, \mathcal{A}, p)$ with $n \geq 3$ and $\mathcal{E} = \emptyset$ is GIWR in $\mathbb{R}^d$ if and only if the weak rigidity matrix $R_W(p)$ has rank $dn - (d^2+d+2)/2$.*

Proof. For $d = 2$ case, the theorem was proved in Theorem 3.1 in [21]. We now prove it for $d = 3$ case.

From Lemmas 2 and 3, when $\mathcal{E} \neq \emptyset$, rank $(R_W(p)) = dn - d(d+1)/2$ if and only if Null $(R_W(p)) = \text{span}(L_R)$. Note that $(\mathbb{1}_n \otimes I_d)$ and $(I_n \otimes J_i)p$, $i \in \{1,2,3\}$ in L_R correspond to a rigid-body translation and a rigid-body rotation of the entire framework, respectively. Therefore, for the case of $\mathcal{E} \neq \emptyset$, the theorem directly follows from Definition 7.

Similarly, when $\mathcal{E} = \emptyset$, rank $(R_W(p)) = dn - (d^2+d+2)/2$ if and only if Null $(R_W(p)) = \text{span}(L_N)$. Since $(\mathbb{1}_n \otimes I_d)$, $(I_n \otimes J_i)p$, $i \in \{1,2,3\}$ and p in L_N correspond to a rigid-body translation, a rigid-body rotation and a scaling of the entire framework, respectively, the remainder of the theorem for the $\mathcal{E} = \emptyset$ case directly follows from Definition 7. □

Remark 2. Comparison with the relevant publications: As stated in Lemmas 1 and 2, the trivial infinitesimal motions in terms of R_W correspond to translations, rotations and scalings when considering no distance constraint whereas those motions related to $\hat{R}_w$ correspond to only a subset of the motions, i.e., translations and rotations without scaling motions, where $\hat{R}_w$ denotes the rigidity matrix introduced in [20]. This difference is due to the fact that, in our work, inner products of inter-agent relative bearings, i.e., cosines of angles, are regarded as angle constraints whereas inner products of inter-agent relative positions are considered as angle information in [20]. This fact can be checked from Lemma 3.6 in [20]. Therefore, the type-1 weak rigidity theory is distinct from our work.

In [29], the angle rigidity theory is introduced, which is a similar concept to the weak rigidity theory in this paper when no distance constraint is considered. The main difference between the work [29] and our work is that we deal with not only 2-dimensional cases but also 3-dimensional cases whereas the paper [29] only studies 2-dimensional cases. In addition, we explore global exponential convergence of 3-agent formations whereas the paper [29] does not. Therefore, our work can include the work in [29].

3.3. Relationship between distance rigidity and GWR

This subsection shows that the proposed theory, i.e., weak rigidity theory, is necessary for the distance rigidity theory [5–8]. First, let us denote a conventional framework without an angle set by $(\mathcal{G}, p)$. We then reach the following result.

Proposition 1. *If a conventional framework $(\mathcal{G}, p)$ is distance rigid, globally distance rigid and infinitesimally distance rigid in $\mathbb{R}^d$, then the framework $(\mathcal{G}, \mathcal{A}, p)$ is GWR, globally GWR and GIWR in $\mathbb{R}^d$, respectively.*

Proof. First, the assumption that $(\mathcal{G}, p)$ is distance rigid means that there exists a neighborhood $\bar{\mathcal{B}}_p \subseteq \mathbb{R}^{dn}$ of p such that $(\mathcal{G}, q)$, $q \in \bar{\mathcal{B}}_p$, equivalent to $(\mathcal{G}, p)$ is congruent to $(\mathcal{G}, p)$ [5]. Then, since the rigid shape of $(\mathcal{G}, p)$ is locally determined, it is obvious that $(\mathcal{G}, \mathcal{A}, p)$ is strongly equivalent and congruent to $(\mathcal{G}, \mathcal{A}, q)$, $q \in \bar{\mathcal{B}}_p$ for any $\mathcal{A}$. Therefore, $(\mathcal{G}, \mathcal{A}, p)$ is GWR from Definition 4. In the same way, it can be shown that global distance rigidity implies global GWR.

Next, consider the distance rigidity matrix R_D defined as $R_D(p) = \frac{1}{2}\frac{\partial \mathbf{D}}{\partial p}$. If $(\mathcal{G}, p)$ is infinitesimally distance rigid in $\mathbb{R}^d$, then $R_D(p)$ is of rank $dn - d(d+1)/2$ [6,8]. With this fact, we can observe from the definition $R_W = \begin{bmatrix} \frac{\partial \mathbf{D}}{\partial p} \\ \frac{\partial \mathbf{A}}{\partial p} \end{bmatrix}$ that there exists a nonzero $(dn - d(d+1)/2) \times (dn - d(d+1)/2)$ minor of R_W. Moreover, from Lemma 3, we have that $\mathrm{rank}(R_W(p)) \le dn - d(d+1)/2$ for $\mathcal{E} \neq \emptyset$. Therefore, R_W is of rank $dn - d(d+1)/2$, which implies that $(\mathcal{G}, \mathcal{A}, p)$ is GIWR from Theorem 1. □

Due to the angle constraints, the GWR theory is not sufficient for the distance rigidity theory. The concept of 'weak' is induced from the fact that the GWR theory is a weaker condition than the conventional distance rigidity theory.

3.4. Generic property

In this subsection, we show that both GWR and GIWR are generic properties. First, we define two smooth manifolds as two sets $\mathcal{M}$ and $\mathcal{M}'$ composed of points congruent to p and proportionally congruent to p, respectively. If the affine span of the configuration p is $\mathbb{R}^d$ (or equivalently p does not lie on any hyperplane in $\mathbb{R}^d$), then $\mathcal{M}$ is $d(d+1)/2$-dimensional and $\mathcal{M}'$ is $(d^2+d+2)/2$-dimensional, because $\mathcal{M}$ arises from the $d(d-1)/2$- and d-dimensional manifold of rotations and translations of $\mathbb{R}^d$, respectively, and $\mathcal{M}'$ arises from $d(d-1)/2$-, d- and 1-dimensional manifold of rotations, translations and scalings of $\mathbb{R}^d$, respectively.

With the smooth map $F_W : \chi \to \mathbb{R}^{m+w}$ for some properly defined $\chi \subset \mathbb{R}^{dn}$, let $r = \max\{\mathrm{rank}(\frac{\partial F_W}{\partial p}) \mid p \in \mathbb{R}^{dn}\}$. Then $p \in \mathbb{R}^{dn}$ is a *regular point* of F_W if $\mathrm{rank}(\frac{\partial F_W}{\partial p}) = r$, and a *singular point* otherwise. With reference to Proposition 2 in [5], if p is a regular point of F_W then there exists a neighborhood $\mathcal{B}_p$ of p such that $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ is a $(dn - r)$-dimensional smooth manifold.

If $p_1, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^d$ when $\mathcal{E} \neq \emptyset$ then it follows from Lemma 3 that

$$\mathrm{rank}(\frac{\partial F_W}{\partial p}) = dn - \mathrm{Null}(\frac{\partial F_W}{\partial p}) \le dn - d(d+1)/2. \tag{17}$$

Moreover, if $p_1, \ldots, p_n$ do not lie on any hyperplane in $\mathbb{R}^d$ when $\mathcal{E} = \emptyset$ then, from Lemma 3, we have that

$$\mathrm{rank}(\frac{\partial F_W}{\partial p}) = dn - \mathrm{Null}(\frac{\partial F_W}{\partial p}) \le dn - (d^2+d+2)/2. \tag{18}$$

In particular, we have that if p is a regular point of F_W then $\mathrm{rank}(R_W(p)) = dn - d(d+1)/2$ in case of $\mathcal{E} \neq \emptyset$ or $\mathrm{rank}(R_W(p)) = dn - (d^2+d+2)/2$ in case of $\mathcal{E} = \emptyset$. We then have the following lemma.

Lemma 4. *Suppose that p is a regular point of F_W and the affine span of $p_1, \ldots, p_n$ is $\mathbb{R}^d$. A framework $(\mathcal{G}, \mathcal{A}, p)$ with $\mathcal{E} \neq \emptyset$ is GWR in $\mathbb{R}^d$ if and only if $\mathrm{rank}(R_W(p)) = dn - d(d+1)/2$. In addition, a framework $(\mathcal{G}, \mathcal{A}, p)$ with $\mathcal{E} = \emptyset$ is GWR in $\mathbb{R}^d$ if and only if $\mathrm{rank}(R_W(p)) = dn - (d^2+d+2)/2$.*

Proof. Let us consider the case of $\mathcal{E} \neq \emptyset$. We have the fact that $R_W(p)$ has the maximum rank, i.e., $\mathrm{rank}(R_W(p)) = dn - d(d+1)/2$. Then, $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ is $d(d+1)/2$-dimensional. Thus, $\mathcal{M}$ and $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ have the same dimension, which implies that the two sets agree near p. Consequently, $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ is the set of $q \in \mathbb{R}^{dn}$ such that $(\mathcal{G}, \mathcal{A}, q)$, $q \in \mathcal{B}_p$, is strongly equivalent to $(\mathcal{G}, \mathcal{A}, p)$, and $\mathcal{M}$ is the set of $q \in \mathbb{R}^{dn}$ such that q is congruent to p. Therefore, $(\mathcal{G}, \mathcal{A}, p)$ is GWR in $\mathbb{R}^d$ as defined in Definition 4.

Similarly, when $\mathcal{E} = \emptyset$, $R_W(p)$ is of the maximum rank, i.e., $\mathrm{rank}(R_W(p)) = dn - (d^2+d+2)/2$. Therefore, $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ is $(d^2+d+2)/2$-dimensional. Two sets $\mathcal{M}'$ and $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ have the same dimension, and this implies that the two sets agree close to p. Consequently, $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ is the set of $q \in \mathbb{R}^{dn}$ such that $(\mathcal{G}, \mathcal{A}, q)$, $q \in \mathcal{B}_p$, is angle equivalent to $(\mathcal{G}, \mathcal{A}, p)$, and $\mathcal{M}'$ is the set of $q \in \mathbb{R}^{dn}$ such that q is proportionally congruent to p. Therefore, $(\mathcal{G}, \mathcal{A}, p)$ is GWR in $\mathbb{R}^d$ as defined in Definition 4.

If $(\mathcal{G}, \mathcal{A}, p)$ is GWR in $\mathbb{R}^d$, then $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ and $\mathcal{M}$ are coincident near p, which implies that $F_W^{-1}(F_W(p)) \cap \mathcal{B}_p$ and $\mathcal{M}$ have the same dimension and $\mathrm{rank}(R_W(p)) = r = dn - d(d+1)/2$ when $\mathcal{E} \neq \emptyset$ (resp. $\mathrm{rank}(R_W(p)) = r = dn - (d^2+d+2)/2$ when $\mathcal{E} = \emptyset$). Hence, we can conclude that the framework $(\mathcal{G}, \mathcal{A}, p)$ with $\mathcal{E} \neq \emptyset$ (resp. $\mathcal{E} = \emptyset$) is GWR in $\mathbb{R}^d$ if and only if $\mathrm{rank}(R_W(p)) = dn - d(d+1)/2$ (resp. $\mathrm{rank}(R_W(p)) = dn - (d^2+d+2)/2$). □

In general, a *generic point* introduced in [31] is used to derive a generic property; however, the notion of the generic point cannot be applied to our work since it cannot describe an equation involving angle constraints in a polynomial form. Thus, in this paper, we do not make use of the notion of the generic point. We next provide the following result to explore a relationship between GWR and GIWR

Proposition 2 (*Relationship between GWR and GIWR*)**.** *Suppose a framework $(\mathcal{G}, \mathcal{A}, p)$, $p = [p_1^\top, \ldots, p_n^\top]^\top \in \mathbb{R}^{dn}$, is in $\mathbb{R}^d$ and the affine span of $p_1, \ldots, p_n$ is $\mathbb{R}^d$. Then, the framework $(\mathcal{G}, \mathcal{A}, p)$ is GIWR in $\mathbb{R}^d$ if and only if p is a regular point of F_W and $(\mathcal{G}, \mathcal{A}, p)$ is GWR in $\mathbb{R}^d$.*

Proof. If a framework $(\mathcal{G}, \mathcal{A}, p)$ is GIWR, then it follows from Theorem 1 that $R_W(p)$ is of rank $dn - d(d+1)/2$ or $dn - (d^2+d+2)/2$, and thus p is a regular point. Moreover, with reference to the proof of Lemma 4, we have that $(\mathcal{G}, \mathcal{A}, p)$ is GWR in $\mathbb{R}^d$.

If p is a regular point of F_W and $(\mathcal{G}, \mathcal{A}, p)$ is GWR in $\mathbb{R}^d$, then $R_W(p)$ has the max rank, i.e., $dn - d(d+1)/2$ or $dn - (d^2+d+2)/2$, from the proof of Lemma 4, which implies that the framework $(\mathcal{G}, \mathcal{A}, p)$ is GIWR from Theorem 1. □

We finally have the following result which shows that both GWR and GIWR for a framework are generic properties.

Proposition 3 (*Generic Property*)**.** *If a framework $(\mathcal{G}, \mathcal{A}, p)$ in $\mathbb{R}^d$ for a regular point p of F_W is GWR (resp. GIWR), then $(\mathcal{G}, \mathcal{A}, q)$ in $\mathbb{R}^d$ for any regular point q of F_W is GWR (resp. GIWR).*

Proof. First, if $(\mathcal{G}, \mathcal{A}, p)$ is GIWR in $\mathbb{R}^d$, then $\mathrm{rank}(R_W(p))$ is equal to $dn - d(d+1)/2$ or $dn - (d^2+d+2)/2$. Moreover, it is clear that $(\mathcal{G}, \mathcal{A}, q)$ is also GIWR in $\mathbb{R}^d$ since q is a regular point and it holds that $R_W(q) = R_W(p)$.

Next, if a framework $(\mathcal{G}, \mathcal{A}, p)$ is GWR and p is a regular point of F_W in $\mathbb{R}^d$, then the framework $(\mathcal{G}, \mathcal{A}, p)$ is GIWR in $\mathbb{R}^d$ from Proposition 2. Moreover, $(\mathcal{G}, \mathcal{A}, q)$ is also GIWR, which implies that $(\mathcal{G}, \mathcal{A}, q)$ is GWR from Proposition 2. □

4. Application to formation control: local convergence of n-agent formations in $\mathbb{R}^d$

We now apply the GWR theory to formation control problems. In this section, we particularly explore local stability on n-agent formations in $\mathbb{R}^d$. This section aims to show local stability for minimally GIWR formations, and for non-minimally GIWR formations, where 'local' means 'close to the desired formation'. In distributed multi-agent systems, the gradient flow law [9,18,32,33] is a popular approach, and we make use of the gradient flow approach to stabilize rigid formation shapes in this paper. We first rigorously define the concept of the minimally GIWR formation as follows.

Definition 8 (*Minimally GIWR*)**.** A framework $(\mathcal{G}, \mathcal{A}, p)$ is *minimally GIWR* in $\mathbb{R}^d$ if the framework $(\mathcal{G}, \mathcal{A}, p)$ is GIWR in $\mathbb{R}^d$ and if no single distance or angle constraint can be removed without losing its GIWR.

It is remarkable that if $(\mathcal{G}, \mathcal{A}, p)$ is minimally GIWR in $\mathbb{R}^d$ then rank(R_W) is exactly equal to the number of edge and angle constraints in the case of $\mathcal{E} \neq \emptyset$ (or only angle constraints in the case of $\mathcal{E} = \emptyset$), i.e., rank$(R_W) = m + w$.

4.1. Equations of motion based on gradient flow approach

We assume that each agent is governed by a single integrator, i.e.,

$$\frac{d}{dt}p_i = \dot{p}_i = u_i,\ i \in \mathcal{V}, \tag{19}$$

where time $t \in [0, \infty)$, and u_i is a control input. Any entries in u_i can be expressed by the relative position vectors of neighbors if a gradient flow law is employed. Note our formation control system makes use of the relative positions of neighbors as sensing variables, and the inter-agent distances and angles of neighbors as control variables.

We define the following two column vectors composed of $\|z_g\|^2$ and A_h:

$$\begin{aligned} d_c(p) &= \left[\ldots, \|z_{g_{ij}}\|^2, \ldots\right]^\top_{(i,j)\in\mathcal{E}}, \\ c_c(p) &= \left[\ldots, A_{h_{kij}}, \ldots\right]^\top_{(k,i,j)\in\mathcal{A}}. \end{aligned} \tag{20}$$

Similarly, d_c^* and c_c^* are defined as

$$d_c^* = \left[\ldots, \|z_g^*\|^2, \ldots\right]^\top,\ c_c^* = \left[\ldots, A_h^*, \ldots\right]^\top, \tag{21}$$

where $\|z_g^*\|^2$ and A_h^* denote the desired values of $\|z_g\|^2$ and A_h, respectively, and both of them are constants. With the above definitions, an error vector is defined as follows:

$$e(p) = \left[d_c(p)^\top c_c(p)^\top\right]^\top - \left[d_c^{*\top} c_c^{*\top}\right]^\top. \tag{22}$$

The simple gradient flow law is employed to analyze a formation control system as follows:

$$\dot{p} = u = -\left(\nabla\left(\frac{1}{2}e^\top(p)e(p)\right)\right)^\top. \tag{23}$$

The control law can be expressed as

$$\begin{aligned} \dot{p} = u &= -\left(\nabla\left(\frac{1}{2}e^\top(p)e(p)\right)\right)^\top = -R_W^\top(p)e(p) \\ &= -\begin{bmatrix} s_1^\top & s_2^\top & \cdots & s_n^\top \end{bmatrix}^\top = -(E(p)\otimes I_d)p \end{aligned} \tag{24}$$

for $s_i \in \mathbb{R}^d$, $i \in \{1, \ldots, n\}$ and $E(p) \in \mathbb{R}^{n\times n}$. In $E(p)$, $[E(p)]_{ij}$ is an element at row i and column j and $[E(p)]_{ij}$ is the coefficient of the vector p_j in s_i. According to the structure of (24), we can observe that the matrix $E(p)$ is symmetric (see an example (12) in [21]). The formation control system (24) is Lipschitz continuous since the system is continuously differentiable, which implies that the solution of (24) exists globally. With (24), we have the following error dynamics:

$$\dot{e} = \frac{\partial e}{\partial p}\dot{p} = R_W(p)\dot{p} = -R_W(p)R_W^\top(p)e. \tag{25}$$

The controller for agent k in (24) can be written by

$$\begin{aligned} \dot{p}_k = &\underbrace{-2\sum_{j\in\mathcal{N}_k^d}\left(\|z_{kj}\|^2 - \|z_{kj}^*\|^2\right)(p_k - p_j)}_{(j,k)\in\mathcal{E}} \\ &\underbrace{-\sum_{i,j\in\mathcal{N}_k^a}\left(\cos\theta_{ij}^k - \cos\left(\theta_{ij}^k\right)^*\right)\left(\frac{\partial}{\partial p_k}\cos\theta_{ij}^k\right)^\top}_{(k,i,j)\in\mathcal{A}} \\ &\underbrace{-\sum_{j,k\in\mathcal{N}_i^a}\left(\cos\theta_{jk}^i - \cos\left(\theta_{jk}^i\right)^*\right)\left(\frac{\partial}{\partial p_k}\cos\theta_{jk}^i\right)^\top}_{\text{if } \exists(i,j,k)\in\mathcal{A}}, \end{aligned} \tag{26}$$

where $\|z_{kj}^*\|$ and $\left(\theta_{ij}^k\right)^*$ are the desired values for $\|z_{kj}\|$ and θ_{ij}^k, respectively, and $\mathcal{N}_k^d = \{j \in \mathcal{V} \mid (j,k) \in \mathcal{E}\}$ and $\mathcal{N}_k^a = \{i,j \in \mathcal{V} \mid (k,i,j) \in \mathcal{A}\}$ denote the neighbor sets for agent k related to distance and angle constraints, respectively. Therefore, it is clear that the system is a distributed system since each agent requires only local information. Moreover, according to the control system (26), we need to define the following assumption for a sensing topology.

Assumption 1. The sensing graph is characterized by an undirected graph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ and agent k can measure relative position vectors in terms of its neighbor set $\mathcal{N}_k^s$, where $\mathcal{V}_s = \mathcal{V}$, $\mathcal{E}_s = \{(i,j), (i,k), (j,k) \mid (i,j) \in \mathcal{E} \vee (k,i,j) \in \mathcal{A}\}$ and $\mathcal{N}_k^s = \{j \in \mathcal{V} \mid (j,k) \in \mathcal{E}_s\}$.

The following result will be useful for next analysis, which shows that if a differential equation $\dot{X}(t) = f(t, X)$ satisfies the following result then the rank of the solution $X(t)$ is constant for all $t \geq 0$ and $\dot{X}(t)$ is said to be *rank-preserving*.

Lemma 5 ([34, Lemma 2])**.** *Let $A(t) \in \mathbb{R}^{M\times M}$ and $B(t) \in \mathbb{R}^{N\times N}$ be a continuous time-varying family of matrices. Then, the following differential equation*

$$\dot{X}(t) = A(t)X(t) + X(t)B(t),\ X(0) \in \mathbb{R}^{M\times N} \tag{27}$$

is rank-preserving.

We next show some properties of the formation control system with the gradient flow approach.

Lemma 6. *Under the gradient flow law, the formation control system designed in (24) has the following properties:*

(i) *The controller is distributed.*
(ii) *The controller and measurement for each agent are independent of any global coordinates. That is, only the local coordinate system for each agent is required to measure relative positions and to implement the control signals.*
(iii) *The centroid* $p^o = \frac{1}{n}\sum_{i=1}^n p_i$ *is stationary. In the case of* $\mathcal{E} = \emptyset$, *the centroid* p^o *and the scale* $p^s = \sqrt{\frac{1}{n}\sum_{i=1}^n \|p_i - p^o\|^2}$ *are both invariant for all* $t \geq 0$.
(iv) *Denote* $C_p = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \in \mathbb{R}^{d\times n}$. *Then,* rank $(C_p(0))$ = rank $(C_p(t))$ *for all time* $t \geq 0$. *Moreover, if* C_p *is of full row rank, then all of* $p_i, \forall i \in \{1, \ldots, n\}$ *do not lie on a hyperplane. On the other hand, if* C_p *is not of full row rank, then there exists a hyperplane containing all* $p_i, \forall i \in \{1, 2, \ldots, n\}$.
(v) *[Collision avoidance for the case of* $\mathcal{E} = \emptyset$*] Let* $p^* = [p_1^{*\top}, \ldots, p_n^{*\top}]^\top \in \mathbb{R}^{dn}$ *denote the desired configuration. Then, it is guaranteed that* $\|p_i(t) - p_j(t)\| > \zeta$ *for all* $t \geq 0$ *and* $i, j \in \mathcal{V}$ *if* $\|p_i^* - p_j^*\| - \sqrt{n}\|p(0) - (\mathbb{1}_n \otimes p^o)\| - \sum_{l=1}^n \|p^o - p_l^*\| > \zeta$ *for* $\zeta > 0$.
(vi) *If a framework* $(\mathcal{G}, \mathcal{A}, p(0))$ *with* $n = d+1$ *vertices is minimally GIWR in* $\mathbb{R}^d$ *and* $C_p(0)$ *is of full row rank, then* $(\mathcal{G}, \mathcal{A}, p(t))$ *is minimally GIWR in* $\mathbb{R}^d$ *for all* $t \geq 0$, *i.e.,* rank $(R_W(p(0)))$ = rank $(R_W(p(t)))$ *for all* $t \geq 0$.

Proof. (i) This property is obvious from (24).

(ii) This property is proved in a similar way to Lemma 4 in [35]. First, let us denote a measurement in a global coordinate system by $(\cdot)^g$. Observe the fact that there exists a rotation matrix $Q_k \in \mathbb{R}^{d\times d}$ such that $p_j = Q_k p_j^g + v$, where v denotes a translation vector. Then, we can express (26) in terms of the global coordinate system as follows:

$$\begin{aligned}
\dot{p}_k^g =& u_k^g \\
=& Q_k^{-1} u_k \\
=& -2Q_k^{-1} \sum_{j\in\mathcal{N}_k^d} \left(\|z_{kj}\|^2 - \|z_{kj}^*\|^2\right)^g Q_k z_{kj}^g \\
& - Q_k^{-1} \sum_{i,j\in\mathcal{N}_k^a} \left(\cos\theta_{ij}^k - \cos\left(\theta_{ij}^k\right)^*\right)^g Q_k \left(\frac{\partial}{\partial p_k}\cos\theta_{ij}^k\right)^{g\top} \\
& \underbrace{- Q_k^{-1} \sum_{j,k\in\mathcal{N}_i^a} \left(\cos\theta_{jk}^i - \cos\left(\theta_{jk}^i\right)^*\right)^g Q_k \left(\frac{\partial}{\partial p_k}\cos\theta_{jk}^i\right)^{g\top}}_{\text{for } \exists(i,j,k)\in\mathcal{A}} \\
=& -2\sum_{j\in\mathcal{N}_k^d} \left(\|z_{kj}\|^2 - \|z_{kj}^*\|^2\right)^g z_{kj}^g \\
& - \sum_{i,j\in\mathcal{N}_k^a} \left(\cos\theta_{ij}^k - \cos\left(\theta_{ij}^k\right)^*\right)^g \left(\frac{\partial}{\partial p_k}\cos\theta_{ij}^k\right)^{g\top} \\
& \underbrace{- \sum_{j,k\in\mathcal{N}_i^a} \left(\cos\theta_{jk}^i - \cos\left(\theta_{jk}^i\right)^*\right)^g \left(\frac{\partial}{\partial p_k}\cos\theta_{jk}^i\right)^{g\top}}_{\text{for } \exists(i,j,k)\in\mathcal{A}}, \qquad (28)
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\frac{\partial}{\partial p_k}\cos(\theta_{ij}^k) \\
&= \frac{\partial}{\partial p_k}\frac{z_{ki}^\top}{\|z_{ki}\|}\frac{z_{kj}}{\|z_{kj}\|} \\
&= \frac{z_{kj}^\top}{\|z_{kj}\|}\frac{1}{\|z_{ki}\|}\left(I_d - \frac{z_{ki}z_{ki}^\top}{\|z_{ki}\|^2}\right) + \frac{z_{ki}^\top}{\|z_{ki}\|}\frac{1}{\|z_{kj}\|}\left(I_d - \frac{z_{kj}z_{kj}^\top}{\|z_{kj}\|^2}\right) \\
&= \frac{z_{kj}^{g\top}}{\|z_{kj}^g\|}Q_k^{-1}\frac{1}{\|z_{ki}^g\|}\left(I_d - Q_k\frac{z_{ki}^g}{\|z_{ki}^g\|}\frac{z_{ki}^{g\top}}{\|z_{ki}^g\|}Q_k^{-1}\right) \\
&\quad + \frac{z_{ki}^{g\top}}{\|z_{ki}^g\|}Q_k^{-1}\frac{1}{\|z_{kj}^g\|}\left(I_d - Q_k\frac{z_{kj}^g}{\|z_{kj}^g\|}\frac{z_{kj}^{g\top}}{\|z_{kj}^g\|}Q_k^{-1}\right) \\
&= \left(\frac{\partial}{\partial p_k}\cos\theta_{ij}^k\right)^g Q_k^{-1}, \qquad (29)
\end{aligned}$$

In the same way as the above result, it also holds that $\frac{\partial}{\partial p_k}\cos(\theta_{jk}^i) = \left(\frac{\partial}{\partial p_k}\cos\theta_{jk}^i\right)^g Q_k^{-1}$. Thus, we conclude the statement.

(iii) Since $p^o = \frac{1}{n}\sum_{i=1}^n p_i = \frac{1}{n}(\mathbb{1}_n \otimes I_d)^\top p \in \mathbb{R}^d$, the following time derivative holds:

$$\begin{aligned}
\dot{p}^o &= \frac{1}{n}(\mathbb{1}_n \otimes I_d)^\top \dot{p} \\
&= -\frac{1}{n}(\mathbb{1}_n \otimes I_d)^\top R_W^\top(p)e(p) \\
&= -\frac{1}{n}\left(\begin{bmatrix}\frac{\partial \mathbf{D}}{\partial z'} \\ \frac{\partial \mathbf{A}}{\partial z'}\end{bmatrix}\bar{H}'(\mathbb{1}_n \otimes I_d)\right)^\top e(p) \qquad (30)
\end{aligned}$$

Since span$(\mathbb{1}_n \otimes I_d) \subseteq$ Null$(\bar{H}') \subseteq$ Null $(R_W(p))$, $R_W(p)(\mathbb{1}_n \otimes I_d) = 0$ and this implies that $\dot{p}^o = 0$. Moreover, it also holds that $\dot{p}^o = 0$ for the case of $\mathcal{E} = \emptyset$.

In the case of $\mathcal{E} = \emptyset$, there is no constraint for the scale of the given framework. Note it holds that $p^s = \sqrt{\frac{1}{n}\sum_{i=1}^n \|p_i - p^o\|^2} = \|p - \mathbb{1}_n \otimes p^o\|/\sqrt{n}$. With the fact that $\dot{p}^o = 0$, we have

$$\dot{p}^s = \frac{1}{\sqrt{n}}\frac{(p - \mathbb{1}_n \otimes p^o)^\top}{\|p - \mathbb{1}_n \otimes p^o\|}\dot{p}. \qquad (31)$$

It holds that $p^\top\dot{p} = -(R_W(p)p)^\top e(p) = 0$ and $(\mathbb{1}_n \otimes p^o)^\top\dot{p} = -(R_W(p)(\mathbb{1}_n \otimes p^o))^\top e(p) = 0$ since span$(p) \subseteq$ Null(R_W) and span$(\mathbb{1}_n \otimes p^o) \subseteq$ Null$(\bar{H}') \subseteq$ Null $(R_W(p))$. Therefore, $\dot{p}^s = 0$. Hence, the statement is proved.

(iv) Since $\dot{p}(t) = -(E(p) \otimes I_d)p(t)$, the vector differential equation can be expressed as the following matrix differential equation.

$$\dot{C}_p(t) = -C_p(t)E^\top(p(t)) \in \mathbb{R}^{d\times n}. \qquad (32)$$

From Lemma 5, the matrix differential equation (32) is rank-preserving for any finite time $t \geq 0$.

If C_p is not of full row rank, then there exists a nontrivial solution x such that $C_p^\top x = 0$. This implies that $p_1^\top x = p_2^\top x = \cdots = p_n^\top x = 0$ and $(p_i^\top - p_j^\top)x = z_{ij}^\top x = 0$ for all $i, j \in \mathcal{V}$ and $i \neq j$, which means that all of vectors z_{ij} are orthogonal to the vector x and further all of vectors z_{ij} lie on a hyperplane. Hence, there exists a hyperplane containing all $p_i, \forall i \in \{1, 2, \ldots, n\}$ if C_p is not of full row rank.

(v) For any $i, j \in \mathcal{V}$ and $t \geq 0$, we have the following equation

$$\begin{aligned}
\|p_i(t) - p_j(t)\| &= \|\left(p_i(t) - p_i^*\right) - \left(p_j(t) - p_j^*\right) + \left(p_i^* - p_j^*\right)\| \\
&\geq \|p_i^* - p_j^*\| - \|p_i(t) - p_i^*\| - \|p_j(t) - p_j^*\| \\
&\geq \|p_i^* - p_j^*\| - \sum_{l=1}^n \|p_l(t) - p_l^*\|, \qquad (33)
\end{aligned}$$

where

$$\begin{aligned}
&\|p_i^* - p_j^*\| - \sum_{l=1}^n \|p_l(t) - p_l^*\| \\
&= \|p_i^* - p_j^*\| - \sum_{l=1}^n \|\left(p_l(t) - p^o\right) + \left(p^o - p_l^*\right)\| \\
&\geq \|p_i^* - p_j^*\| - \sum_{l=1}^n \|p_l(t) - p^o\| - \sum_{l=1}^n \|p^o - p_l^*\|
\end{aligned}$$

$$\geq \|p_i^* - p_j^*\| - \sqrt{n}\|p(t) - (\mathbb{1}_n \otimes p^o)\| - \sum_{l=1}^{n} \|p^o - p_l^*\|. \quad (34)$$

In the above inequality (34), it holds that $\sqrt{n}\|p(t)-(\mathbb{1}_n \otimes p^o)\| \geq \sum_{l=1}^{n} \|p_l(t) - p^o\|$ by using the following inequality for positive real numbers $x_1, \ldots, x_n$.

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n}. \quad (35)$$

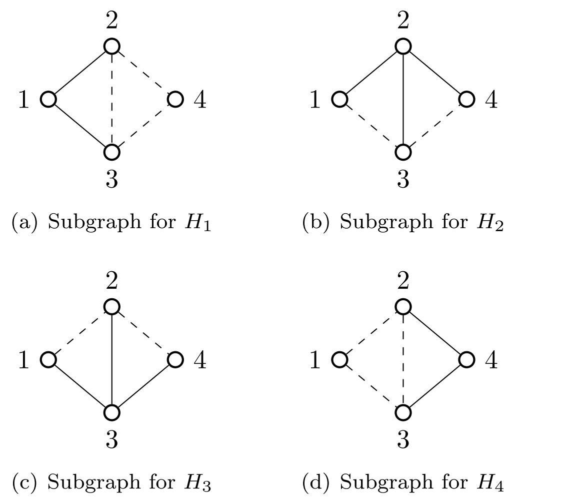
Since $\|p(t)-(\mathbb{1}_n \otimes p^o)\|$ has the similar form to p^s as given in the proof of Lemma 6–(iii), the time derivative of $\|p(t) - (\mathbb{1}_n \otimes p^o)\|$ equals zero, and this follows that $\|p(t)-(\mathbb{1}_n \otimes p^o)\|$ is invariant for all $t \geq 0$. Here p^o is also invariant. Thus, if $\|p_i^* - p_j^*\| - \sqrt{n}\|p(0) - (\mathbb{1}_n \otimes p^o)\| - \sum_{l=1}^{n} \|p^o - p_l^*\|$ is greater than ζ for $\zeta > 0$ at $t = 0$, then $\|p_i(t) - p_j(t)\|$ is also greater than ζ for all $t \geq 0$.

(vi) This proof is motivated by Theorem 4.4 in [20]. Let us first consider $R_W(p(0)) = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_\sigma \end{bmatrix}^\top = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$ at $t = 0$, where $\mathbf{r}_i \in \mathbb{R}^{dn}$, $i \in \{1, \ldots, \sigma\}$, $\mathbf{c}_j \in \mathbb{R}^{\sigma \times d}$, $j \in \{1, \ldots, n\}$, and $\sigma = m + w$. We define a set $\mathcal{N}'_l$ of neighbors of vertex l as $\mathcal{N}'_l = \{i, j \in \mathcal{V} \mid (l, i) \in \mathcal{E} \vee (l, i, j) \in \mathcal{A}\}$. If a framework $(\mathcal{G}, \mathcal{A}, p)$ with $n = d + 1$ vertices is minimally GIWR, then each agent has exactly d neighbors, i.e., $|\mathcal{N}'_l| = n - 1 = d$.

Let a framework $(\mathcal{G}, \mathcal{A}, p(0))$ with $n = d + 1$ vertices be minimally GIWR, and let $C_p(0)$ be of full row rank. Suppose that the framework $(\mathcal{G}, \mathcal{A}, p(t^*))$ is not GIWR at specific time $t^* > 0$. Then, $R_W(p(t^*))$ does not have full row rank, and further there exists a nonzero vector $\tau = \begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_\sigma \end{bmatrix}^\top \in \mathbb{R}^\sigma$ such that $\tau^\top R_W(p(t^*)) = \tau_1\mathbf{r}_1^\top + \tau_2\mathbf{r}_2^\top + \cdots + \tau_\sigma\mathbf{r}_\sigma^\top = 0$ (or equivalently $\tau_1\mathbf{r}_1 + \tau_2\mathbf{r}_2 + \cdots + \tau_\sigma\mathbf{r}_\sigma = 0$). Since $\tau^\top R_W(p(t^*)) = \tau^\top \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{bmatrix} = 0$, $\tau^\top \mathbf{c}_l = \tau^\top \frac{\partial F_W}{\partial p_l} = 0$ for all $l \in \{1, 2, \ldots, n\}$. Note that each entry for the weak rigidity matrix R_W is composed of inter-neighbor relative position vectors from a framework $(\mathcal{G}, \mathcal{A}, p)$. From the fact that $\frac{\partial F_W}{\partial p_l}$ consists of $z'^\top_{lk}(t^*)$ for $k \in \mathcal{N}'_l$ and $\tau^\top \mathbf{c}_l = 0$, there must exist at least one case from $l = 1$ to $l = n$ such that $z'^\top_{lk}(t^*)$ for $k \in \mathcal{N}'_l$ are linearly dependent.

With $|\mathcal{N}'_l| = n - 1 = d$, we can denote an oriented incidence matrix H_l associated with the vertex l (for example, see Fig. 4), where $H_l \in \mathbb{R}^{d \times (d+1)}$ for all $l \in \{1, \ldots, n\}$. We define a matrix $E_l(t^*)$ composed of $z'^\top_{lk}(t^*)$ for $k \in \mathcal{N}'_l$ as $E_l(t^*) = H_l C_p^\top(t^*) \in \mathbb{R}^{d \times d}$. We can state $E_l(t^*)$ as $E_l(t^*) = \begin{bmatrix} \cdots, z'_{lk}(t^*), \ldots \end{bmatrix}^\top$. Consider $E_l(t^*)x = 0$ for any nontrivial $x \in \mathbb{R}^d$ and $l \in \{1, \ldots, n\}$, then either the equality $C_p^\top(t^*)x = 0$ or the equality $z'^\top_{ij}x = 0, \forall i, j \in \mathcal{V}'$ holds. The equality $z'^\top_{ij}x = 0, \forall i, j \in \mathcal{V}'$ means that all of vectors z'_{ij} are orthogonal to the vector x, and further all of vectors z'_{ij} lie on a hyperplane. Thus, the equality $z'^\top_{ij}x = 0$ cannot hold as proved in Lemma 6–(iv). The equality $C_p^\top(t^*)x = 0$ cannot also hold since $C_p(t^*)$ has the full row rank for all $t \geq 0$ as proved in Lemma 6–(iv). Hence, Null $(E_l(t)) = \emptyset$ and the rank of $E_l(t^*)$ equals d. However, there exists at least one case such that $z'^\top_{lk}(t^*)$ for $k \in \mathcal{N}'_l$ are linearly dependent, and this follows that rank $(E_l(t^*)) < d$. This conflicts with rank $(E_l(t^*)) = d$. Hence, we can conclude that $(\mathcal{G}, \mathcal{A}, p(t))$ is minimally GIWR for all $t \geq 0$ if $(\mathcal{G}, \mathcal{A}, p(0))$ with $n = d + 1$ vertices is minimally GIWR and $C_p(0)$ is of full row rank. □

Assumption 2. In formation control problems addressed in this paper, it is assumed that any two agents at the initial time are sufficiently far from each other to not make any collision between agents with reference to Lemma 6–(v).



(a) Subgraph for H_1 (b) Subgraph for H_2

(c) Subgraph for H_3 (d) Subgraph for H_4

Fig. 4. Example of subgraphs for H_l when $n = 4$. The dashed lines indicate the removed edges. The graphs have the same vertex set but do not have the same edge set.

4.2. Exponential stability of minimally GIWR formations with n agents in $\mathbb{R}^d$

We first explore the stability of minimally GIWR formations with n agents in $\mathbb{R}^d$. In this subsection, we assume that the desired formation is minimally GIWR, which is relaxed in the next subsection.

Theorem 2. *Suppose that the desired formation is minimally GIWR and the control system (24) follows Assumption 1. If any initial formation is close to the desired formation, then the error system (25) has an exponentially stable equilibrium at the origin, and the initial formation locally exponentially converges to the desired formation shape.*

Proof. We first define the potential function $V(e)$ as $V(e) = \frac{1}{2}e^\top e$ which is also the Lyapunov function candidate. We also define a sub-level set Ψ as $\Psi = \{e \mid V(e) \leq \epsilon\}$ for $\epsilon > 0$ such that all formations in the set Ψ are minimally GIWR close to the desired formation.

With Eq. (25), the derivative of $V(e)$ along a trajectory of e is calculated as

$$\dot{V}(e) = e^\top \dot{e} = -e^\top R_W(e) R_W^\top(e) e = -\|R_W^\top(e) e\|^2. \quad (36)$$

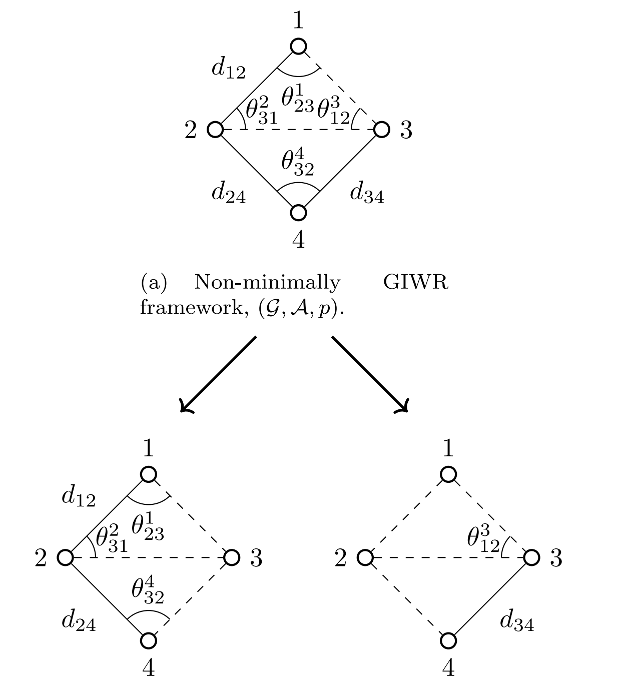
Since the formation in the set Ψ is minimally GIWR, the weak rigidity matrix has the full row rank. Therefore, since rank $\left(R_W(e)R_W^\top(e)\right)$ = rank $(R_W(e))$, $R_W(e)R_W^\top(e)$ is of full rank and $R_w(e)R_w^\top(e)$ is positive definite (all eigenvalues of $R_w(e)R_w^\top(e)$ are positive). Moreover, this implies

$$\dot{V}(e) \leq -\lambda\|e\|^2, \quad (37)$$

where λ denotes the minimum eigenvalue of $R_w(e)R_w^\top(e)$. The inequality (37) indicates that $\dot{V} < 0$ for $e \in \Psi \setminus \{0\}$. Thus, the origin of the error system (25) is asymptotically stable near the desired formation. Also, since $V = \frac{1}{2}e^\top e$, the following inequality holds.

$$\dot{V}(e) \leq -2\lambda V(e), \quad (38)$$

and it follows by Gronwall–Bellman Inequality [36, Lemma A.1] that $V(e(t)) \leq V(e(0))\exp(-2\lambda t)$. Therefore, the error system (25) has an exponentially stable equilibrium at the origin, and the solution of (24) exists and is finite as $t \to \infty$. By the



(a) Non-minimally GIWR framework, $(\mathcal{G},\mathcal{A},p)$.

(b) Minimally GIWR sub-framework, $(\bar{\mathcal{G}},\bar{\mathcal{A}},p)$.

(c) Remaining sub-framework, $(\tilde{\mathcal{G}},\tilde{\mathcal{A}},p)$.

Fig. 5. Example of framework decomposition of a non-minimally GIWR framework. The dashed lines indicate virtual edges which do not belong to $\mathcal{E}$, $\bar{\mathcal{E}}$ and $\tilde{\mathcal{E}}$. Distance and angle constraints are denoted by $d_{ij}, (i,j) \in \mathcal{E}$ and $\theta_{ij}^k, (k,i,j) \in \mathcal{A}$, respectively.

above result, the control law (24) guarantees that p exponentially converges to a fixed point. The initial formation in the set Ψ is close to the desired formation. Hence, the initial formation locally exponentially converges to the desired formation shape. $\square$

4.3. Stability on non-minimally GIWR formations with n agents in $\mathbb{R}^d$

In this subsection, we explore the stability in the case of non-minimally GIWR formation systems with n agents in $\mathbb{R}^d$. To this end, we make use of a linearization approach of perturbed systems motivated by [3,37].

We denote a minimally GIWR sub-framework induced from $(\mathcal{G},\mathcal{A},p)$ by $(\bar{\mathcal{G}},\bar{\mathcal{A}},p)$, where $\bar{\mathcal{G}} = (\mathcal{V},\bar{\mathcal{E}})$. We also denote the remaining part of $(\mathcal{G},\mathcal{A},p)$ except $(\bar{\mathcal{G}},\bar{\mathcal{A}},p)$ by $(\tilde{\mathcal{G}},\tilde{\mathcal{A}},p)$, where $\tilde{\mathcal{G}} = (\mathcal{V},\tilde{\mathcal{E}})$, $\tilde{\mathcal{E}} = \mathcal{E} \setminus \bar{\mathcal{E}}$ and $\tilde{\mathcal{A}} = \mathcal{A} \setminus \bar{\mathcal{A}}$ (see an example in Fig. 5). Let σ denote the sum of cardinalities of edges and angles, i.e., $\sigma = m + w$. Then, $\bar{\sigma}$ and $\tilde{\sigma}$ are defined as $\bar{\sigma} = |\bar{\mathcal{E}}| + |\bar{\mathcal{A}}| = \bar{m} + \bar{w} = dn - d(d+1)/2$ (or $dn - (d^2+d+2)/2$ when $\mathcal{E} = \emptyset$) and $\tilde{\sigma} = |\tilde{\mathcal{E}}| + |\tilde{\mathcal{A}}| = \tilde{m} + \tilde{w} = \sigma - \bar{\sigma}$, respectively. Moreover, we denote the sub-vector $\bar{e} \in \mathbb{R}^{\bar{\sigma}}$ whose entries are those entries in e corresponding to edges and angles in $(\bar{\mathcal{G}},\bar{\mathcal{A}},p)$, and $\tilde{e} \in \mathbb{R}^{\tilde{\sigma}}$ whose entries are those entries in e corresponding to edges and angles in $(\tilde{\mathcal{G}},\tilde{\mathcal{A}},p)$. We denote the permutation matrix $\mathbf{P} = \begin{bmatrix}\bar{\mathbf{P}}^\top & \tilde{\mathbf{P}}^\top\end{bmatrix}$ such that $\begin{bmatrix}\bar{e} & \tilde{e}\end{bmatrix}^\top = \mathbf{P}^\top e$ or equivalently $\bar{e} = \bar{\mathbf{P}}e$ and $\tilde{e} = \tilde{\mathbf{P}}e$, where $\mathbf{P} \in \mathbb{R}^{\sigma\times\sigma}$, $\bar{\mathbf{P}} \in \mathbb{R}^{\bar{\sigma}\times\sigma}$ and $\tilde{\mathbf{P}} \in \mathbb{R}^{\tilde{\sigma}\times\sigma}$. The permutation matrix has properties such that $\bar{\mathbf{P}}\bar{\mathbf{P}}^\top = I_{\bar{\sigma}\times\bar{\sigma}}$, $\tilde{\mathbf{P}}\tilde{\mathbf{P}}^\top = I_{\tilde{\sigma}\times\tilde{\sigma}}$, $\bar{\mathbf{P}}\tilde{\mathbf{P}}^\top = 0_{\bar{\sigma}\times\tilde{\sigma}}$, $\bar{\mathbf{P}}^\top\bar{\mathbf{P}} + \tilde{\mathbf{P}}^\top\tilde{\mathbf{P}} = I_{\sigma\times\sigma}$ and $e = \bar{\mathbf{P}}^\top\bar{e} + \tilde{\mathbf{P}}^\top\tilde{e}$. We now show that $\tilde{e}$ is a function of $\bar{e}$ locally.

Lemma 7. *Let a framework $(\mathcal{G},\mathcal{A},q)$ be the desired formation, and non-minimally GIWR. Then, there (locally) exists a smooth function $f : \bar{e}(q) \to \mathbb{R}^{(\sigma-\bar{\sigma})}$ such that $\tilde{e}(q) = f(\bar{e}(q))$ close to $(\mathcal{G},\mathcal{A},q)$. Furthermore, it holds that $f(\bar{e}) = 0$ if and only if $\bar{e} = 0$.*

Proof. This proof is motivated by Proposition 1 in [37]. (i) For the 2-dimensional case, we first denote a rotation matrix $S(\mathbf{x})$ such that $S(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}\begin{bmatrix} x_2 & -x_1 \\ x_1 & x_2\end{bmatrix}$ for a nonzero vector $\mathbf{x} = \begin{bmatrix}x_1 & x_2\end{bmatrix}^\top \in \mathbb{R}^2$. The equality $S(\mathbf{x})\mathbf{x} = \begin{bmatrix}0 & \|\mathbf{x}\|\end{bmatrix}^\top$ always holds. We denote a vector $\varsigma : p \to \mathbb{R}^{\bar{\sigma}}$ with $\bar{\sigma} = 2n-3$ when $\mathcal{E} \neq \emptyset$ in $\mathbb{R}^2$ such as:

$$\varsigma(p) = \begin{bmatrix}\|z_{21}\| & (S(z_{21})z_{31})^\top & \cdots & (S(z_{21})z_{n1})^\top\end{bmatrix}^\top. \quad (39)$$

Since the rotation matrix does not change a magnitude of a vector, we see that $\|z_{j1}\|^2 = \|S(z_{21})z_{j1}\|^2$ and $\|z_{ij}\|^2 = \|S(z_{21})z_{i1} - S(z_{21})z_{j1}\|^2$, and further $\varsigma(p)$ includes all information on the relative vectors $z_{21}, z_{31}, \ldots, z_{n1}$. Thus, any entry in $\tilde{e}$ is composed of entries in $\varsigma(p)$. Moreover, there exists a function $\tilde{f}_e : \mathbb{R}^{\bar{\sigma}} \to \mathbb{R}^{(\sigma-\bar{\sigma})}$ such that $\tilde{e} = \tilde{f}_e(\varsigma(p))$. Similarly, there exists a function $\bar{f}_e : \mathbb{R}^{\bar{\sigma}} \to \mathbb{R}^{\bar{\sigma}}$ such that $\bar{e} = \bar{f}_e(\varsigma(p))$.

In the same way, for the case of $\mathcal{E} = \emptyset$ in $\mathbb{R}^2$, we can define a vector $\varsigma : p \to \mathbb{R}^{\bar{\sigma}}$ with $\bar{\sigma} = 2n-4$ such that

$$\varsigma(p) = \begin{bmatrix}(S(z_{21})z_{31})^\top & (S(z_{21})z_{41})^\top & \cdots & (S(z_{21})z_{n1})^\top\end{bmatrix}^\top. \quad (40)$$

Then, with the fact in [38, Lemma 11], it is obvious that there exist $\tilde{e} = \tilde{f}_e(\varsigma(p))$ and $\bar{e} = \bar{f}_e(\varsigma(p))$.

The derivative of $\bar{e}$ at q, i.e., $\left.\frac{\partial\bar{e}(p)}{\partial p}\right|_{p=q}$ is the weak rigidity matrix of $(\bar{\mathcal{G}},\bar{\mathcal{A}},q)$. Then, rank $\left(\left.\frac{\partial\bar{e}(p)}{\partial p}\right|_{p=q}\right) = \bar{\sigma}$ since $(\bar{\mathcal{G}},\bar{\mathcal{A}},q)$ is minimally GIWR. Thus, with the fact that $\left.\frac{\partial\bar{e}(p)}{\partial p}\right|_{p=q} = \left.\frac{\bar{f}_e(\varsigma(p))}{\partial\varsigma(p)}\frac{\partial\varsigma(p)}{\partial p}\right|_{p=q}$ from $\bar{e} = \bar{f}_e(\varsigma(p))$, it holds that rank $\left(\left.\frac{\bar{f}_e(\varsigma(p))}{\partial\varsigma(p)}\right|_{p=q}\right) \geq \bar{\sigma}$ by the rank property. Since $\left.\frac{\bar{f}_e(\varsigma(p))}{\partial\varsigma(p)}\right|_{p=q}$ is an $\bar{\sigma}\times\bar{\sigma}$ matrix, we can see that $\left.\frac{\bar{f}_e(\varsigma(p))}{\partial\varsigma(p)}\right|_{p=q}$ is of full rank and $\left.\frac{\bar{f}_e(\varsigma(p))}{\partial\varsigma(p)}\right|_{p=q}$ is nonsingular. Hence, from the inverse function theorem, there is an open set $\mathcal{W} \subset \mathbb{R}^{\bar{\sigma}}$ containing $\varsigma(q)$ such that $\bar{f}_e$ has a smooth inverse $\bar{f}_e^{-1} : \bar{f}_e(\mathcal{W}) \to \mathcal{W}$. Then, the following equality holds.

$$\bar{f}_e^{-1}(\bar{f}_e(\varsigma(p))) = \varsigma(p),\ \varsigma(p) \in \mathcal{W}, \quad (41)$$

which implies that $\bar{f}_e^{-1}(\bar{f}_e(\varsigma(p))) = \bar{f}_e^{-1}(\bar{e}) = \varsigma(p)$. Since $\tilde{e} = \tilde{f}_e(\varsigma(p))$, the equality $\tilde{e} = \tilde{f}_e(\bar{f}_e^{-1}(\bar{e}))$ holds. Therefore, we can say that there exists a smooth function $f : \bar{e}(q) \to \mathbb{R}^{(\sigma-\bar{\sigma})}$ such that $\tilde{e}(q) = f(\bar{e}(q))$ close to $(\mathcal{G},\mathcal{A},q)$. In addition, since $\tilde{\mathbf{P}}e = \tilde{e} = \tilde{f}_e\left(\bar{f}_e^{-1}(\bar{e})\right) = \tilde{f}_e\left(\bar{f}_e^{-1}\left(\bar{\mathbf{P}}e\right)\right) = f\left(\bar{\mathbf{P}}e\right)$ and $e = 0$ at the desired formation $(\mathcal{G},\mathcal{A},q)$, it holds that $f(0) = 0$.

(ii) For the 3-dimensional case, let us consider rotation matrices $S_{x_1}(\mathbf{x})$ and $S_{x_2}(\mathbf{x})$ rotating a vector $\mathbf{x} = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}^\top \in \mathbb{R}^3$ about x_1 and x_2 axes into x_1x_3-plane and x_1x_2-plane, respectively, as follows:

$$S_{x_1}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{x_3}{\sqrt{x_2^2+x_3^2}} & -\frac{x_2}{\sqrt{x_2^2+x_3^2}} \\ 0 & \frac{x_2}{\sqrt{x_2^2+x_3^2}} & \frac{x_3}{\sqrt{x_2^2+x_3^2}} \end{bmatrix},$$

$$S_{x_2}(\mathbf{x}) = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_3^2}} & 0 & \frac{x_3}{\sqrt{x_1^2+x_3^2}} \\ 0 & 1 & 0 \\ -\frac{x_3}{\sqrt{x_1^2+x_3^2}} & 0 & \frac{x_1}{\sqrt{x_1^2+x_3^2}} \end{bmatrix}. \quad (42)$$

We then have

$$S_{x_1}(z_{21})z_{21} = \begin{bmatrix} z_{21}^{(1)} & 0 & \frac{\left(z_{21}^{(2)}\right)^2+\left(z_{21}^{(3)}\right)^2}{\sqrt{\left(z_{21}^{(2)}\right)^2+\left(z_{21}^{(3)}\right)^2}} \end{bmatrix}^\top = \bar{z}_{21}, \tag{43}$$

$$S_{x_2}(\bar{z}_{21})\bar{z}_{21} = \begin{bmatrix} \|z_{21}\| & 0 & 0 \end{bmatrix}^\top = \check{\bar{z}}_{21}, \tag{44}$$

where $z_{21} = \begin{bmatrix} z_{21}^{(1)} & z_{21}^{(2)} & z_{21}^{(3)} \end{bmatrix}^\top \in \mathbb{R}^3$. We also have

$$S_{x_1}(\check{\bar{z}}_{31})\check{\bar{z}}_{21} = \begin{bmatrix} \|z_{21}\| & 0 & 0 \end{bmatrix}^\top, \tag{45}$$

$$S_{x_1}(\check{\bar{z}}_{31})\check{\bar{z}}_{31} = \begin{bmatrix} \check{\bar{z}}_{31}^{(1)} & 0 & \frac{\left(\check{\bar{z}}_{31}^{(2)}\right)^2+\left(\check{\bar{z}}_{31}^{(3)}\right)^2}{\sqrt{\left(\check{\bar{z}}_{31}^{(2)}\right)^2+\left(\check{\bar{z}}_{31}^{(3)}\right)^2}} \end{bmatrix}^\top, \tag{46}$$

where $\check{\bar{z}}_{31} = S_{x_2}(\bar{z}_{21})S_{x_1}(z_{21})z_{31} = \begin{bmatrix} \check{\bar{z}}_{31}^{(1)} & \check{\bar{z}}_{31}^{(2)} & \check{\bar{z}}_{31}^{(3)} \end{bmatrix}^\top \in \mathbb{R}^3$. With the facts of (45) and (46), we can denote a vector $\varsigma : p \to \mathbb{R}^{\bar{\sigma}}$ with $\bar{\sigma} = 3n-6$ when $\mathcal{E} \neq \emptyset$ in $\mathbb{R}^3$ such that

$$\varsigma(p) =$$
$$\begin{bmatrix} \|z_{21}\| & \check{\bar{z}}_{31}^{(1)} & \frac{\left(\check{\bar{z}}_{31}^{(2)}\right)^2+\left(\check{\bar{z}}_{31}^{(3)}\right)^2}{\sqrt{\left(\check{\bar{z}}_{31}^{(2)}\right)^2+\left(\check{\bar{z}}_{31}^{(3)}\right)^2}} & \left(\bar{S}z_{41}\right)^\top & \cdots & \left(\bar{S}z_{n1}\right)^\top \end{bmatrix}^\top, \tag{47}$$

where $\bar{S} = S_{x_1}(\check{\bar{z}}_{31})S_{x_2}(\bar{z}_{21})S_{x_1}(z_{21})$. $\varsigma(p)$ includes all information on the relative vectors $z_{21}, z_{31}, \ldots, z_{n1}$. Since the rotation matrices do not change a magnitude of a vector, any entry in $\tilde{e}$ is a function composed of entries in $\varsigma(p)$, and further there exists a function $\tilde{f}_e : \mathbb{R}^{\bar{\sigma}} \to \mathbb{R}^{(\sigma-\bar{\sigma})}$ such that $\tilde{e} = \tilde{f}_e(\varsigma(p))$. Moreover, there exists a function $\bar{f}_e : \mathbb{R}^{\bar{\sigma}} \to \mathbb{R}^{\bar{\sigma}}$ such that $\bar{e} = \bar{f}_e(\varsigma(p))$. In the same manner, for the case of $\mathcal{E} = \emptyset$ in $\mathbb{R}^3$, we can define a vector $\varsigma : p \to \mathbb{R}^{\bar{\sigma}}$ with $\bar{\sigma} = 3n-7$ such that

$$\varsigma(p) = \begin{bmatrix} \check{\bar{z}}_{31}^{(1)} & \frac{\left(\check{\bar{z}}_{31}^{(2)}\right)^2+\left(\check{\bar{z}}_{31}^{(3)}\right)^2}{\sqrt{\left(\check{\bar{z}}_{31}^{(2)}\right)^2+\left(\check{\bar{z}}_{31}^{(3)}\right)^2}} & \left(\bar{S}z_{41}\right)^\top & \cdots & \left(\bar{S}z_{n1}\right)^\top \end{bmatrix}^\top. \tag{48}$$

Then, with reference to [38, Lemma 11], there exist $\tilde{e} = \tilde{f}_e(\varsigma(p))$ and $\bar{e} = \bar{f}_e(\varsigma(p))$. The rest of this proof is proved in the same way as the 2-dimensional case. □

We denote $\bar{R}_W \in \mathbb{R}^{\bar{\sigma}\times dn}$ as the weak rigidity matrix for the sub-framework $(\bar{\mathcal{G}}, \bar{\mathcal{A}}, p)$, and $\tilde{R}_W \in \mathbb{R}^{\tilde{\sigma}\times dn}$ as the weak rigidity matrix for the sub-framework $(\tilde{\mathcal{G}}, \tilde{\mathcal{A}}, p)$. Then, it holds that $\bar{R}_W = \bar{\mathbf{P}}R_W$ and $\tilde{R}_W = \tilde{\mathbf{P}}R_W$. From the fact that $\bar{e} = \bar{\mathbf{P}}e$ and $e = \bar{\mathbf{P}}^\top\bar{e} + \tilde{\mathbf{P}}^\top\tilde{e}$, we have

$$\begin{aligned} \dot{\bar{e}} = \bar{\mathbf{P}}\dot{e} = \bar{\mathbf{P}}\frac{\partial e}{\partial p}\dot{p} &= -\bar{\mathbf{P}}R_W R_W^\top e \\ &= -\bar{\mathbf{P}}R_W R_W^\top(\bar{\mathbf{P}}^\top\bar{e} + \tilde{\mathbf{P}}^\top\tilde{e}) \\ &= -\bar{R}_W\bar{R}_W^\top\bar{e} - \bar{R}_W\tilde{R}_W^\top\tilde{e}. \end{aligned} \tag{49}$$

From Lemma 7, the equality (49) can be rewritten as

$$\dot{\bar{e}} = -\bar{R}_W\bar{R}_W^\top\bar{e} - \bar{R}_W\tilde{R}_W^\top f(\bar{e}), \tag{50}$$

which locally holds only around the desired formation. It also holds that $\tilde{R}_W = \frac{\partial\tilde{e}}{\partial p} = \frac{\partial\tilde{e}}{\partial\bar{e}}\frac{\partial\bar{e}}{\partial p} = \frac{\partial f(\bar{e})}{\partial\bar{e}}\frac{\partial\bar{e}}{\partial p} = \frac{\partial f(\bar{e})}{\partial\bar{e}}\bar{R}_W$. Therefore, we can consider the error system (50) as a perturbed system. We then reach the following theorem.

Theorem 3. *Under the gradient flow law (24) and Assumption 1, the perturbed error system (50) for a non-minimally GIWR formation has an exponential stable equilibrium at the origin.*

Proof. Note that $\tilde{R}_W = \frac{\partial\tilde{e}}{\partial p} = \frac{\partial f}{\partial\bar{e}}\frac{\partial\bar{e}}{\partial p} = F\bar{R}_W$, where $F = \frac{\partial f}{\partial\bar{e}}\ g_k(e_k)$. We define a neighborhood set Ψ around $\bar{e} = 0$ as $\Psi = \{\bar{e} \in \mathbb{R}^{\bar{\sigma}} \mid \|\bar{e}\|^2 < \epsilon\}$ for $\epsilon > 0$. Then, the remainder of this proof is similar to Theorem 3 in [3]. □

5. Application to formation control: almost global convergence of 3-agent formations in $\mathbb{R}^2$

This section aims to provide analysis for almost global stability on special cases of minimally GIWR 3-agent formations in $\mathbb{R}^2$. In this section, we also use the control system (24) as discussed in Section 4.1. We first classify all equilibrium points to explore the stability of the system (24) with a set $\mathcal{P}$ composed of all equilibrium points defined as $\mathcal{P} = \{p \in \mathbb{R}^{2n} \mid R_W^\top e = 0\}$ as follows:

$$\mathcal{P}^* = \{p \in \mathbb{R}^{2n} \mid e = 0\}, \tag{51}$$

$$\mathcal{P}_i = \{p \in \mathbb{R}^{2n} \mid R_W^\top e = 0, e \neq 0\}, \tag{52}$$

where $\mathcal{P}^*$ and $\mathcal{P}_i$ denote the sets for desired equilibria and incorrect equilibria, respectively. Both of $\mathcal{P}^*$ and $\mathcal{P}_i$ constitute the set of all equilibria, i.e., $\mathcal{P} = \mathcal{P}^* \cup \mathcal{P}_i$. An equilibrium point $\bar{p} = [\bar{p}_1^\top, \ldots, \bar{p}_n^\top]^\top \in \mathbb{R}^{2n}$ is called *incorrect equilibrium point* if $\bar{p}$ belongs to $\mathcal{P}_i$.

5.1. Analysis of the incorrect equilibria

We show in this subsection that the system (24) at any incorrect equilibrium point $\bar{p}$ is unstable. We first explore what cases occur at the incorrect equilibria.

Lemma 8. *In the case of three-agent formations, incorrect equilibria take place only when the three agents are collinear.*

Proof. From the viewpoint of a minimally GIWR formation composed of three agents, there are only three formation cases: the first one is a formation with one angle constraint and two distance constraints; the second one is that with two angle constraints and one distance constraint; the third one is that with only two angle constraints. Each example for the three cases is illustrated in Figs. 1(a)–1(c), respectively.

Let $\mathcal{N}'_l$ denote a set of neighbors of vertex l by $\mathcal{N}'_l = \{i, j \in \mathcal{V} \mid (l,i) \in \mathcal{E} \vee (l,i,j) \in \mathcal{A}\}$. If a framework $(\mathcal{G}, \mathcal{A}, p)$ with $n = 3$ vertices is minimally GIWR, then each agent has exactly two neighbors, i.e., $|\mathcal{N}'_l| = 2$. In the weak rigidity matrix R_W, all elements are composed of inter-neighbor relative position vectors, i.e., $\frac{\partial F_W}{\partial p_l}$ consists of ${z'}_{lk_1}^\top$ and ${z'}_{lk_2}^\top$ for $k_1, k_2 \in \mathcal{N}'_l$. Thus, at the incorrect equilibria, the following form holds:

$${z'}_{lk_1}^\top = c_l {z'}_{lk_2}^\top,\ k_1, k_2 \in \mathcal{N}'_l, \tag{53}$$

where $c_l \in \mathbb{R}$ is a coefficient. This implies that incorrect equilibria take place only when the three agents are collinear for 3-agent formations in $\mathbb{R}^2$.

We next show an example with a formation in Fig. 1(a). For the case of the formation with one angle constraint and two distance constraints as shown in Fig. 1(a), Eq. (24) can be written as

$$\dot{p}_1 = -2z_{12}e_{12} - 2z_{13}e_{13} - \alpha^\top e_{23}^1, \tag{54a}$$

$$\dot{p}_2 = 2z_{12}e_{12} - \beta^\top e_{23}^1, \tag{54b}$$

$$\dot{p}_3 = 2z_{13}e_{13} - \gamma^\top e_{23}^1, \tag{54c}$$

where $e_{ij} = \|z_{g_{ij}}\|^2 - \|z_{g_{ij}}^*\|^2$, $(i,j) \in \mathcal{E}$, $e_{23}^1 = A_{h_{123}} - A_{h_{123}}^*$, $\alpha = \frac{\partial}{\partial p_1}\cos\theta_{23}^1$, $\beta = \frac{\partial}{\partial p_2}\cos\theta_{23}^1$ and $\gamma = \frac{\partial}{\partial p_3}\cos\theta_{23}^1$. In the incorrect equilibrium set $\mathcal{P}_i$, Eq. (54c) is calculated as

$$z_{12} = \left.\left(\frac{\|z_{12}\|}{\|z_{13}\|}\cos\theta_{23}^1 - 2\|z_{12}\|\|z_{13}\|\frac{e_{13}}{e_{23}^1}\right)z_{13}\right|_{p\in\mathcal{P}_i} \tag{55}$$

It follows from (55) that p_1, p_2 and p_3 must be collinear. Eqs. (54a) and (54b) also give us similar results. Therefore, the three agents must be collinear. The formation shape of the three agents falls

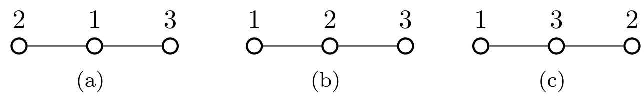


Fig. 6. Three formation forms which can occur at the incorrect equilibria.

into one of three cases as depicted in Fig. 6. Two cases illustrated in Figs. 1(b) and 1(c) also give us similar results to the case of Fig. 1(a). □

Next, to analyze the stability at the incorrect equilibria, we linearize the system (24). One can observe the following negative Jacobian $J(p)$ of the system (24) with respect to p:

$$\begin{aligned}
J(p) &= -\frac{\partial}{\partial p}\dot{p} \\
&= R_W(p)^\top R_W(p) + E(p)\otimes I_2 \\
&\quad + \sum_{(k,i,j)\in\mathcal{A}} e_{A_h}\Big((I_3\otimes p_1)\frac{\partial}{\partial p}C_1 + (I_3\otimes p_2)\frac{\partial}{\partial p}C_2 \\
&\quad + (I_3\otimes p_3)\frac{\partial}{\partial p}C_3\Big),
\end{aligned} \tag{56}$$

where $p = \begin{bmatrix} p_1^\top & p_2^\top & p_3^\top \end{bmatrix}^\top \in \mathbb{R}^6$, $e_{A_h} = A_{h_{kij}} - A^*_{h_{kij}}$, and $C_l \in \mathbb{R}^3$ for $l \in \{1,2,3\}$ denotes a vector composed of entries of lth column associated with e_{A_h} in $E(p)$ (see an example (17) in [21]). If $J(p)$ has at least one negative eigenvalue at the incorrect equilibrium point $\bar{p}$, then the system at $\bar{p}$ is unstable. In order to check this fact, we first reorder columns of $J(p)$, which does not have an effect on any eigenvalue of $J(p)$. We make use of a permutation matrix T which reorders columns of matrix such that

$$\begin{aligned}
R_W T &= \begin{bmatrix} R_x & R_y \end{bmatrix} = \bar{R}, \\
P_l T &= \begin{bmatrix} P_{lx} & P_{ly} \end{bmatrix} = \bar{P}_l, \\
\frac{\partial}{\partial p}C_l T &= \begin{bmatrix} C_{lx} & C_{ly} \end{bmatrix} = \bar{C}_l,
\end{aligned} \tag{57}$$

where $P_l = (I_3 \otimes p_l^\top) \in \mathbb{R}^{3\times 6}$ for $l \in \{1,2,3\}$. In (57), $R_u \in \mathbb{R}^{\sigma\times 3}$, $P_{lu} \in \mathbb{R}^{3\times 3}$ and $C_{lu} \in \mathbb{R}^{3\times 3}$ for $u = x, y$ denote matrices whose columns are composed of the columns of coordinate u in the matrix R_W, P_l and $\frac{\partial}{\partial p}C_l$, respectively. The formation is minimally GIWR, thus $\sigma = 3$. It is remarkable that $TT^\top = I$ holds since T is a permutation matrix. With the permutation matrix T, the permutated matrix $\bar{J}(p)$ is given by

$$\begin{aligned}
\bar{J}(p) &= T^\top J(p) T \\
&= \bar{R}^\top \bar{R} + I_2 \otimes E(p) + \sum_{(k,i,j)\in\mathcal{A}} \left(\bar{P}_1^\top \bar{C}_1 + \bar{P}_2^\top \bar{C}_2 + \bar{P}_3^\top \bar{C}_3\right) e_{A_h} \\
&= \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} \\ \bar{J}_{21} & \bar{J}_{22} \end{bmatrix},
\end{aligned} \tag{58}$$

where

$$\begin{aligned}
\bar{J}_{11} &= R_x^\top R_x + E(p) + \sum_{(k,i,j)\in\mathcal{A}} (P_{1x}C_{1x} + P_{2x}C_{2x} + P_{3x}C_{3x}) e_{A_h}, \\
\bar{J}_{12} &= R_x^\top R_y + \sum_{(k,i,j)\in\mathcal{A}} \left(P_{1x}C_{1y} + P_{2x}C_{2y} + P_{3x}C_{3y}\right) e_{A_h}, \\
\bar{J}_{21} &= R_y^\top R_x + \sum_{(k,i,j)\in\mathcal{A}} \left(P_{1y}C_{1x} + P_{2y}C_{2x} + P_{3y}C_{3x}\right) e_{A_h}, \\
\bar{J}_{22} &= R_y^\top R_y + E(p) + \sum_{(k,i,j)\in\mathcal{A}} (P_{1y}C_{1y} + P_{2y}C_{2y} + P_{3y}C_{3y}) e_{A_h}.
\end{aligned}$$

Note that the stability of an equilibrium point is independent of a rigid-body translation, a rigid-body rotation and a scaling of an entire framework since relative distances and subtended angles only matter. Therefore, without loss of generality, we suppose that $\bar{p}$ lies on the x-axis since they are collinear. Then, we have $R_y = 0$, $P_{1y} = 0$, $C_{1y} = 0$, $P_{2y} = 0$, $C_{2y} = 0$, $P_{3y} = 0$ and $C_{3y} = 0$. Then, $\bar{J}(\bar{p})$ is of the form

$$\bar{J}(\bar{p}) = \begin{bmatrix} \bar{J}_{11}(\bar{p}) & 0 \\ 0 & E(\bar{p}) \end{bmatrix}. \tag{59}$$

The following results show that the system (24) at $\bar{p}$ is unstable.

Lemma 9. *Let $\bar{p}$ be in the incorrect equilibrium set $\mathcal{P}_i$. Then, $E(\bar{p})$ has at least one negative eigenvalue.*

Proof. We first define α, β and γ as $\alpha = \frac{\partial}{\partial p_k}\cos\theta_{ij}^k$, $\beta = \frac{\partial}{\partial p_i}\cos\theta_{ij}^k$ and $\gamma = \frac{\partial}{\partial p_j}\cos\theta_{ij}^k$, and let α_{p_k}, α_{p_i} and α_{p_j} denote coefficients of p_k, p_i and p_j in α, respectively. Similarly, β_{p_k}, β_{p_i}, β_{p_j}, γ_{p_k}, γ_{p_i} and γ_{p_j} are denoted. Then, from the structure of the matrix E, we can have the following equation in case of $\mathcal{E} \neq \emptyset$ for a configuration $\hat{p} = [\hat{p}_1^\top, \ldots, \hat{p}_n^\top]^\top \in \mathbb{R}^{2n}$.

$$\begin{aligned}
&\hat{p}^\top [E(\bar{p})\otimes I_d]\hat{p} \\
&= 2\sum_{(i,j)\in\mathcal{E}} e_{ij}(\bar{p})\|\hat{p}_i - \hat{p}_j\|^2 \\
&\quad + \sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar{p})\big(\hat{p}_k^\top \hat{p}_k \alpha_{\bar{p}_k} + \hat{p}_k^\top \hat{p}_i \alpha_{\bar{p}_i} + \hat{p}_k^\top \hat{p}_j \alpha_{\bar{p}_j} \\
&\quad + \hat{p}_i^\top \hat{p}_k \beta_{\bar{p}_k} + \hat{p}_i^\top \hat{p}_i \beta_{\bar{p}_i} + \hat{p}_i^\top \hat{p}_j \beta_{\bar{p}_j} \\
&\quad + \hat{p}_j^\top \hat{p}_k \gamma_{\bar{p}_k} + \hat{p}_j^\top \hat{p}_i \gamma_{\bar{p}_i} + \hat{p}_j^\top \hat{p}_j \gamma_{\bar{p}_j}\big) \\
&= 2\sum_{(i,j)\in\mathcal{E}} e_{ij}(\bar{p})\|\hat{p}_i - \hat{p}_j\|^2 - \sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar{p})\big(\|\hat{p}_k - \hat{p}_i\|^2 \beta_{\bar{p}_k} \\
&\quad + \|\hat{p}_k - \hat{p}_j\|^2 \alpha_{\bar{p}_j} + \|\hat{p}_i - \hat{p}_j\|^2 \gamma_{\bar{p}_i}\big),
\end{aligned} \tag{60}$$

where $e_{ij}(\bar{p}) = \|z(\bar{p})_{ij}\|^2 - \|z_{ij}^*\|^2$, $e_{A_h}(\bar{p}) = A_{h_{kij}}\big|_{p=\bar{p}} - A^*_{h_{kij}}$,

$$\begin{aligned}
\beta_{\bar{p}_k} &= \frac{-1}{\|\bar{z}_{ki}\|\|\bar{z}_{kj}\|} + \left(\frac{\|\bar{z}_{ki}\|^2 + \|\bar{z}_{kj}\|^2 - \|\bar{z}_{ij}\|^2}{2\|\bar{z}_{ki}\|\|\bar{z}_{kj}\|}\right)\frac{1}{\|\bar{z}_{ki}\|^2}, \\
\alpha_{\bar{p}_j} &= \frac{-1}{\|\bar{z}_{ki}\|\|\bar{z}_{kj}\|} + \left(\frac{\|\bar{z}_{ki}\|^2 + \|\bar{z}_{kj}\|^2 - \|\bar{z}_{ij}\|^2}{2\|\bar{z}_{ki}\|\|\bar{z}_{kj}\|}\right)\frac{1}{\|\bar{z}_{kj}\|^2}, \\
\gamma_{\bar{p}_i} &= \frac{1}{\|\bar{z}_{ki}\|\|\bar{z}_{kj}\|},
\end{aligned}$$

$\bar{z}_{ij} = \bar{p}_i - \bar{p}_j$ and it holds that $\alpha_{\bar{p}_i} = \beta_{\bar{p}_k}$, $\alpha_{\bar{p}_j} = \gamma_{\bar{p}_k}$ and $\beta_{\bar{p}_j} = \gamma_{\bar{p}_i}$, and it also holds that $\alpha_{\bar{p}_k} + \alpha_{\bar{p}_i} + \alpha_{\bar{p}_j} = 0$, $\beta_{\bar{p}_k} + \beta_{\bar{p}_i} + \beta_{\bar{p}_j} = 0$ and $\gamma_{\bar{p}_k} + \gamma_{\bar{p}_i} + \gamma_{\bar{p}_j} = 0$. In the case of $\mathcal{E} = \emptyset$, we have

$$\begin{aligned}
&\hat{p}^\top [E(\bar{p})\otimes I_d]\hat{p} \\
&= -\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar{p})\big(\|\hat{p}_k - \hat{p}_i\|^2 \beta_{\bar{p}_k} + \|\hat{p}_k - \hat{p}_j\|^2 \alpha_{\bar{p}_j} + \|\hat{p}_i - \hat{p}_j\|^2 \gamma_{\bar{p}_i}\big).
\end{aligned} \tag{61}$$

Suppose that $E(\bar{p})$ is positive semidefinite. Then, we have $\hat{p}^\top[E(\bar{p})\otimes I_d]\hat{p} \geq 0$ for any configuration $\hat{p} \in \mathbb{R}^{2n}$. Consider the desired configuration $p^* = [p_1^{*\top}, \ldots, p_n^{*\top}]^\top \in \mathbb{R}^{2n}$ in $\mathcal{P}^*$. With the fact that the equality (60) and $\bar{p}^\top[E(\bar{p})\otimes I_d]\bar{p} = 0$, the following equation holds.

$$\begin{aligned}
&p^{*\top}[E(\bar{p})\otimes I_d]p^* \\
&= p^{*\top}[E(\bar{p})\otimes I_d]p^* - \bar{p}^\top[E(\bar{p})\otimes I_d]\bar{p}
\end{aligned}$$

$$
\begin{aligned}
=&2\sum_{(i,j)\in\mathcal{E}} e_{ij}(\bar p)\|p_i^*-p_j^*\|^2-2\sum_{(i,j)\in\mathcal{E}} e_{ij}(\bar p)\|\bar p_i-\bar p_j\|^2\\
&-\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar p)\left(\|z_{ki}^*\|^2\beta_{\bar p_k}+\|z_{kj}^*\|^2\alpha_{\bar p_j}+\|z_{ij}^*\|^2\gamma_{\bar p_i}\right)\\
&+\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar p)\left(\|\bar z_{ki}\|^2\beta_{\bar p_k}+\|\bar z_{kj}\|^2\alpha_{\bar p_j}+\|\bar z_{ij}\|^2\gamma_{\bar p_i}\right)\\
=&2\sum_{(i,j)\in\mathcal{E}} e_{ij}(\bar p)\|p_i^*-p_j^*\|^2-2\sum_{(i,j)\in\mathcal{E}} e_{ij}(\bar p)\|\bar p_i-\bar p_j\|^2\\
&-\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar p)\left(\|z_{ki}^*\|^2\beta_{\bar p_k}+\|z_{kj}^*\|^2\alpha_{\bar p_j}+\|z_{ij}^*\|^2\gamma_{\bar p_i}\right)\\
&+\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar p)\frac{\|\bar z_{ki}\|^2+\|\bar z_{kj}\|^2-\|\bar z_{ij}\|^2}{2\|\bar z_{ki}\|\|\bar z_{kj}\|}\left(\frac{2\|z_{ki}^*\|\|z_{kj}^*\|}{\|\bar z_{ki}\|\|\bar z_{kj}\|}\right)\\
&-\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar p)\frac{\|\bar z_{ki}\|^2+\|\bar z_{kj}\|^2-\|\bar z_{ij}\|^2}{2\|\bar z_{ki}\|\|\bar z_{kj}\|}\left(\frac{2\|z_{ki}^*\|\|z_{kj}^*\|}{\|\bar z_{ki}\|\|\bar z_{kj}\|}\right)\\
=&-2\sum_{(i,j)\in\mathcal{E}}|e_{ij}(\bar p)|^2-\sum_{(k,i,j)\in\mathcal{A}}|e_{A_h}(\bar p)|^2\left(\frac{2\|z_{ki}^*\|\|z_{kj}^*\|}{\|\bar z_{ki}\|\|\bar z_{kj}\|}\right)\\
&+\sum_{(k,i,j)\in\mathcal{A}} e_{A_h}(\bar p)\frac{\|\bar z_{ki}\|^2+\|\bar z_{kj}\|^2-\|\bar z_{ij}\|^2}{2\|\bar z_{ki}\|\|\bar z_{kj}\|}\left(\frac{2\|z_{ki}^*\|\|z_{kj}^*\|}{\|\bar z_{ki}\|\|\bar z_{kj}\|}\right.\\
&\left.-\frac{\|z_{ki}^*\|^2}{\|\bar z_{ki}\|^2}-\frac{\|z_{kj}^*\|^2}{\|\bar z_{kj}\|^2}\right),
\end{aligned}
\tag{62}
$$

where $z_{ij}^*=p_i^*-p_j^*$ and it holds that $\|\bar z_{ik}\|^2\beta_{\bar p_k}+\|\bar z_{jk}\|^2\alpha_{\bar p_j}+\|\bar z_{ij}\|^2\gamma_{\bar p_i}=0$. It follows from Lemma 8 that the incorrect equilibrium point $\bar p$ lies on the 1-dimensional space. Thus, $\left(\frac{\|\bar z_{ki}\|^2+\|\bar z_{kj}\|^2-\|\bar z_{ij}\|^2}{2\|\bar z_{ki}\|\|\bar z_{kj}\|}\right)^2=\left(\cos\theta_{ij}^k\right)^2\Big|_{p=\bar p}=1$, which implies that

$$
\begin{aligned}
&e_{A_h}(\bar p)\left(\frac{\|\bar z_{ki}\|^2+\|\bar z_{kj}\|^2-\|\bar z_{ij}\|^2}{2\|\bar z_{ki}\|\|\bar z_{kj}\|}\right)\\
&=1-\left(\cos\theta_{ij}^k\right)\big|_{p=p^*}\left(\cos\theta_{ij}^k\right)\big|_{p=\bar p}\ge 0.
\end{aligned}
\tag{63}
$$

Moreover, it holds that $\left(\frac{2\|z_{ki}^*\|\|z_{kj}^*\|}{\|\bar z_{ki}\|\|\bar z_{kj}\|}-\frac{\|z_{ki}^*\|^2}{\|\bar z_{ki}\|^2}-\frac{\|z_{kj}^*\|^2}{\|\bar z_{kj}\|^2}\right)=-\left(\frac{\|z_{ki}^*\|}{\|\bar z_{ki}\|}-\frac{\|z_{kj}^*\|}{\|\bar z_{kj}\|}\right)^2\le 0$. Therefore, we have $p^{*\top}[E(\bar p)\otimes I_d]p^*<0$ when $\mathcal{E}\neq\emptyset$. Similarly, when $\mathcal{E}=\emptyset$, it also holds that $p^{*\top}[E(\bar p)\otimes I_d]p^*<0$. However, this conflicts with $\hat p^\top[E(\bar p)\otimes I_d]\hat p\ge 0$ for any configuration $\hat p$. Hence, we have the statement. □

Theorem 4. *The system (24) at any incorrect equilibrium point $\bar p$ is unstable.*

Proof. Since $\bar J(\bar p)$ is of the form (59), if $E(\bar p)$ has at least one negative eigenvalue then $\bar J(\bar p)$ also has at least one negative eigenvalue. From Lemma 9, we know that $E(\bar p)$ has at least one negative eigenvalue and the matrix $\bar J(\bar p)$ also does. Since eigenvalues of $\bar J(\bar p)$ and $J(\bar p)$ are the same, $J(\bar p)$ also has at least one negative eigenvalue. Hence, the system (24) at any incorrect equilibrium point $\bar p$ is unstable. □

5.2. Almost global stability on 3-agent formation in $\mathbb{R}^2$

This subsection shows that if a configuration p does not belong to $\mathcal{P}_i$ then p does not approach $\mathcal{P}_i$ as time goes on. Finally, this subsection provides the main result of the almost global stability on 3-agent formations in $\mathbb{R}^2$.

Lemma 10. *Let $p(0)$ denote an initial formation. If $p(0)$ given by the gradient flow law (24) does not belong to the set of incorrect equilibria, $\mathcal{P}_i$, then $p(t)$ does not approach $\mathcal{P}_i$ for any time $t\ge 0$.*

Proof. For a 3-agent formation in $\mathbb{R}^2$, an incorrect equilibrium point $\bar p$ always lies on a hyperplane, i.e., $\operatorname{rank}(C_{\bar p}(t))<d$ from Lemma 8. Additionally, the linearized version of the system (24), i.e., negative Jacobian $J(p)$, at an incorrect equilibrium point $\bar p$ has at least one negative eigenvalue from Theorem 4. Hence, this property is proved straightforward by a similar approach to the proof of Theorem 2 in [34]. □

Theorem 5. *Under the control system (24) and Assumption 1, if a framework $(\mathcal{G},\mathcal{A},p(0))$ with $n=3$ is minimally GIWR and $p(0)$ is not in the incorrect equilibrium set $\mathcal{P}_i$ in $\mathbb{R}^2$, then $p(0)$ exponentially converges to a point in the desired equilibrium set $\mathcal{P}^*$.*

Proof. We define a Lyapunov function candidate as $V(e)=\frac{1}{2}e^\top e$. Notice that $V(e)\ge 0$ with $V(e)=0$ if and only if $e=0$ and V is radially unbounded. The time derivative of $V(e)$ along a trajectory of e is calculated as

$$
\dot V=e^\top\dot e=-e^\top R_W R_W^\top e=-\|R_W^\top e\|^2. \tag{64}
$$

We know that $\dot V\le 0$, $\dot V$ is equal to zero if and only if $R_W^\top e=0$. From Theorem 4, Lemma 10 and the assumption that $p(0)\notin\mathcal{P}_i$, it follows that $e\to 0$ asymptotically fast and the error system (25) has an asymptotically stable equilibrium at the origin.

From $p(0)\notin\mathcal{P}_i$, the initial positions do not lie on the 1-dimensional space, i.e., $C_p(0)$ is of full row rank. Then, from Lemma 6–(vi), $\operatorname{rank}(R_W(p(0)))=\operatorname{rank}(R_W(p(t)))$ for all $t\ge 0$ in $\mathbb{R}^d$. It follows from $p(0)\notin\mathcal{P}_i$ and Lemma 6–(vi) that $R_W R_W^\top$ is positive definite for all $t\ge 0$. Henceforth, Eq. (64) satisfies

$$
\dot V\le-\lambda(R_W R_W^\top)\|e\|^2,
$$

where λ denotes the minimum eigenvalue of $R_W R_W^\top$ along this trajectory. Moreover, since $V=\frac{1}{2}e^\top e$, the following inequality holds.

$$
\dot V(e)\le-2\lambda V(e), \tag{65}
$$

and it follows by Gronwall–Bellman Inequality [36, Lemma A.1] that $V(e(t))\le V(e(0))\exp(-2\lambda t)$. Therefore, $e\to 0$ exponentially fast and the error system (25) has an exponentially stable equilibrium at the origin, which in turn implies that $p\to p^*$ for all initial positions outside the set $\mathcal{P}_i$, where p^* is the desired formation. Hence, we conclude that the formation control system (24) almost globally exponentially converges to the desired formation in $\mathcal{P}^*$. □

6. Conclusion

This paper studied the GWR theory and stability for the formation control system based on the GWR theory in the 2- and 3-dimensional spaces. Based on the GWR theory, we can determine a rigid formation shape with a set of pure inter-agent distances and angles. In particular, with using the rank condition of the weak rigidity matrix, we can conveniently examine whether a formation shape is rigid or not. We also showed that both GWR and GIWR for a framework are generic properties, and the GWR theory is necessary for the distance rigidity theory. We then applied the GWR theory to the formation control with the gradient descent flow law. As the first result of its applications, we proved the local exponential stability for GIWR formations in the 2 and 3-dimensional spaces. Finally, for 3-agent formations in the 2-dimensional space, we showed the almost global exponential stability of the formation control system. Readers who are interested in simulation examples on the formation control can refer to the preprint version [38].

As a future work, we first aim to extend the GWR based formation control to GWR based flocking control with the double-integrator model. We expect that a flocking control system based on the GWR theory can be developed in a similar way as the flocking control with the distance rigidity theory [39,40]. We then expect that a rigid cooperative manipulation scheme as studied [41] can be developed with the proposed rigidity theory in this paper. A rigid point set registration [42] is also of our interest as a future work. To the best of our knowledge, a rigid point set registration has been studied with global information and centralized schemes. We expect that the GWR theory can contribute to a distributed scheme for the rigid point set registration with only local information.

CRediT authorship contribution statement

Seong-Ho Kwon: Formal analysis, Writing - original draft, Data curation. **Hyo-Sung Ahn:** Conceptualization, Funding acquisition, Supervision, Validation, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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