Discrete-Time Matrix-Weighted Consensus

Quoc Van Tran, Minh Hoang Trinh, and Hyo-Sung Ahn

***Abstract*—This article investigates discrete-time consensus of multiagent networks over undirected and connected graphs, whose edges are weighted by nonnegative definite matrices, under various scenarios. In particular, we first present consensus protocols for the agents in common networks of symmetric matrix weights with possibly different step sizes and switching network topologies. A special type of matrix-weighted consensus with nonsymmetric matrix weights that can render several consensus control scenarios, such as ones with scaled/rotated updates and affine motion constraints, is also considered. We employ Lyapunov stability theory for discrete-time systems and occasionally utilize convex optimization theory for Lyapunov functions with Lipschitz continuous gradients to show convergence to a consensus of the agents. Finally, simulation results are provided to illustrate the theoretical results.**

***Index Terms*—Discrete-time systems, matrix-weighted consensus, multiagent systems, switching graphs.**

I. Introduction

Reaching a consensus on some local decision states is a crucial task in many problems in networked systems, such as distributed control and estimation [1]–[3], distributed optimization, and machine learning [4], [5]. In this context, each agent in the system holds a local (decision) state, obtains the states of other agents via interagent measurements or communication, and updates its state along the direction of a weighted sum of the relative states to neighboring agents.

Although consensus algorithms with scalar weights have been studied extensively, matrix-weighted consensus has been of particular interest recently. Matrix weights can capture interdependencies or impose cross-coupling constraints on the relative vectors of the agents, which is not achievable if only scalar weights were used. Therefore, systems with matrix weights arise naturally in various disciplines of science and engineering including matrix-weighted consensus/synchronization [6]–[9]; opinion dynamics [10], [11]; and distributed control and estimation [12]–[14]. As opposed to consensus with scalar weights, where a condition on the network connectivity is often sufficient for reaching a consensus, in matrix-weighted cases, a consensus is jointly determined by the network connectivity and the matrix weights [7] (or, equivalently, the matrix-weighted Laplacian). Indeed, the authors in [7] reveal that consensus and clustering phenomena exist naturally in a matrix-weighted consensus protocol, and a sufficient condition for reaching consensus is the existence of a positive spanning tree in the (undirected) graph of the system. Bipartite consensus can be achieved if the matrix-weighted graph is structurally balanced and contains a positive–negative spanning tree, whose edge weights are either positive or negative definite [8]. Recently, the authors in [15] study continuous-time matrix-weighted consensus with time-varying network topologies using the notion of the matrix-weighted integral network. The work provides a necessary and sufficient null space condition on the matrix-weighted Laplacian of the integral network for the multiagent system to achieve average consensus. This null space condition returns to that in [7] for the case of time-invariant graphs. Although for a directed graph (*digraph*) with scalar weights, a bound[1]. on the spectrum of the associated Laplacian, which is diagonally dominant, can be effectively characterized by using the Geršhgorin disk theorem [2], [16], and the extension to the case of matrix-weighted Laplacian is not straightforwardly applicable. As a result, existing results on (continuous-time) matrix-weighted consensus for digraphs are limited to only special graph topologies such as leader–follower networks with directed acyclic graphs [17] and weight-balanced digraphs [18]. Matrix-weighted consensus on more general digraphs is thus still an open problem.

However, the aforementioned works in matrix-weighted consensus have been investigated in continuous-time scenarios. In this work, we attempt to investigate discrete-time matrix-weighted consensus of multiagent systems over undirected graphs under several scenarios. Practical motivations for studying discrete-time matrix-weighted consensus are as follows. Discrete-time algorithms are needed in discrete-time cyber-physical systems in which control and learning algorithms are implemented in digital computers or microcontrollers. From a communication-efficiency perspective, discrete-time algorithms are also more favorable for practical implementations as continuous-time algorithms require an infinite information transmission rate. For this reason, most of the existing distributed optimization and machine learning algorithms, and particularly, those based on (scalar-weighted) consensus, are in a discrete-time setting [4]. Apart from these practical aspects, we

[1]All the eigenvalues (except the simple zero eigenvalue) of the Laplacian are contained in the open right half plane if and only if there exists a directed spanning tree in the corresponding digraph [16, Thm. 2]

Manuscript received February 14, 2021; accepted March 12, 2021. Date of publication March 23, 2021; date of current version December 3, 2021. This work was supported by the National Research Foundation (NRF) of Korea under Grant NRF-2017R1A2B3007034. Recommended by Associate Editor G. Russo. *(Corresponding author: Hyo-Sung Ahn.)*

Quoc Van Tran and Hyo-Sung Ahn are with the School of Mechanical Engineering, Gwangju Institute of Science and Technology, Gwangju 61005, South Korea (e-mail: tranvanquoc@gist.ac.kr; hyosung@ieee.org).

Minh Hoang Trinh is with the Department of Automatic Control, School of Electrical Engineering, Hanoi University of Science and Technology (HUST), Hanoi 11615, Vietnam (e-mail: minhtrinh@ieee.org).

Digital Object Identifier 10.1109/TCNS.2021.3068367

explained further the motivations by elaborating the intuitions and applications of our proposed consensus schemes in the main text of the article. To analyze the convergence of the proposed consensus algorithms, we employ Lyapunov stability theory for discrete-time systems and occasionally utilize convex optimization theory for Lyapunov functions with Lipschitz continuous gradients.

The contribution of this article is three-fold.

1) We first study the discrete-time matrix-weighted consensus of multiagent systems with connected graphs, in which the agents can use the same or different step sizes. Furthermore, asymptotic convergence to the average consensus of the system with time-varying graph topologies is also established, provided that the union of the switching graphs over each successive time interval of the same length contains a positive spanning tree. The use of switching (matrix-weighted) graphs poses a mild assumption as it allows the network topology to be disconnected at any time instant. Unlike [15], the proposed discrete-time consensus scheme with switching graphs requires only a finite information transmission rate and reduces further the amount of exchanged data by a suitable design of the matrix weights (Remark 2).
2) Second, consensus of the agents is examined when each agent i in the system employs the same matrix weight $\boldsymbol{A}_{ij} = \boldsymbol{A}_i$ for every relative state to its neighbor j. This setting is different from those in [7], [8], and [15] which commonly require symmetric matrix weights, i.e., $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}$, where $\boldsymbol{A}_{ij}$ and $\boldsymbol{A}_{ji}$ are the matrix weights associated with two neighboring agents i and j, respectively. Suppose that the interaction graph of the system is connected, the agents' step sizes are sufficiently small, and the matrix weights $\boldsymbol{A}_i$ are (possibly nonsymmetric) positive definite. Then, by utilizing convex optimization theory for Lyapunov functions with Lipschitz continuous gradients, we show that the agents achieve a consensus. Compared with the (continuous-time) consensus scheme with rotation matrix weights in [9], the proposed consensus protocol uses more general (positive definite) matrix weights and is applicable for arbitrary dimensions.
3) Third, as an extension to the preceding case, we consider the case that the matrix weight $\boldsymbol{A}_i$ associated with an agent i can be positive semidefinite. We show that the state vector of agent i is constrained in a linear manifold, whose tangent space spans the column space of $\boldsymbol{A}_i$. Then, it is proven that if the intersection of the agents' subspaces is nonempty and the step sizes of the agents are sufficiently small, the agents still achieve a consensus.

The rest of this article is as follows. Preliminaries and problem formulation are provided in Section II. Section III presents consensus protocols for systems with symmetric matrix weights and possibly time-varying network topologies. Consensus schemes over undirected networks with asymmetric matrix weights are investigated in Sections IV, and V. Simulation results are provided in Section VI and Section VII concludes this article.

Notation: Let $\mathbb{R}^d$ and $\mathbb{C}^d$ be the real and complex d-dimensional spaces, respectively. The set of nonnegative integers is $\mathbb{Z}^+$. The notation $||\cdot||$ denotes the Euclidean norm. Let $\text{diag}(\boldsymbol{A}_1,\ldots,\boldsymbol{A}_n) \in \mathbb{R}^{N\times N}$, $\tilde{}N := \sum_{i=1}^n d_i$, be a block-diagonal matrix constructed from $\boldsymbol{A}_1 \in \mathbb{R}^{d_1\times d_1},\ldots,\boldsymbol{A}_n \in \mathbb{R}^{d_n\times d_n}$. The Cartesian product of $\{\mathcal{X}_i\}_{i=1}^n \subseteq \mathbb{R}^d$ is denoted by $\prod_{i=1}^n \mathcal{X}_i$. The relation $\boldsymbol{A} > 0$ ($\boldsymbol{A} \geq 0$) implies that the matrix $\boldsymbol{A}$ is positive definite (positive semidefinite).

II. Preliminaries and Problem Formulation

A. Matrix Weighted Graph

A *matrix-weighted graph* of a multiagent network is denoted by $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V} = \{1,\ldots,n\}$ denotes the vertex set, and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ denotes the set of edges of $\mathcal{G}$, and $\mathcal{A} = \{\boldsymbol{A}_{ij} \in \mathbb{R}^{d\times d} : (i,j) \in \mathcal{E}, \boldsymbol{A}_{ij} \geq 0\}$. An edge is defined by the ordered pair $e_k = (i,j), i \neq j, k = 1,\ldots,m, m = |\mathcal{E}|$. The graph $\mathcal{G}$ is said to be undirected if $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$, i.e., if j is a neighbor of i, then i is also a neighbor of j. If the graph $\mathcal{G}$ is directed, $(i,j) \in \mathcal{E}$ does not necessarily imply $(j,i) \in \mathcal{E}$. The set of neighboring agents of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. Associate each edge $(i,j) \in \mathcal{E}$ with the matrix weight $\boldsymbol{A}_{ij} \geq 0$, and $\boldsymbol{A}_{ij} = \boldsymbol{0}$ when $(i,j) \notin \mathcal{E}$. An edge (i,j) is *positive definite* (positive semidefinite) if $\boldsymbol{A}_{ij} > 0$ ($\boldsymbol{A}_{ij} \geq 0$ and $\boldsymbol{A}_{ij} \neq \boldsymbol{0}$). The *matrix-weighted adjacency matrix* of $\mathcal{G}$ is given as $\boldsymbol{A} = [\boldsymbol{A}_{ij}] \in \mathbb{R}^{nd\times nd}$.

We define $\boldsymbol{D}_i := \sum_{j\in\mathcal{N}_i} \boldsymbol{A}_{ij}$ and let $\boldsymbol{D} = \text{diag}(\boldsymbol{D}_1,\ldots,\boldsymbol{D}_n)$ be the *degree matrix* of the graph $\mathcal{G}$. Then, the *matrix-weighted Laplacian* is given as $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A} \in \mathbb{R}^{nd\times nd}$. Denote $\boldsymbol{L}^{\text{o}}$ as the *identity-matrix weighted Laplacian* of $\mathcal{G}$ with $\boldsymbol{A}_{ij} = \boldsymbol{I}_d$ $\forall(i,j) \in \mathcal{E}$, and $\boldsymbol{A}_{ij} = \boldsymbol{0}$ otherwise.

When the matrix weights in the graph are *symmetric*, i.e., $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji} \geq 0 \forall(i,j) \in \mathcal{E}$, the following established lemma can be obtained [7].

Lemma 1: The matrix-weighted Laplacian $\boldsymbol{L}$ is symmetric and positive semidefinite, and its null space is given as $\text{null}(\boldsymbol{L}) = \text{span}\{\text{range}(\boldsymbol{1}_n \otimes \boldsymbol{I}_d), \{\boldsymbol{v} = [\boldsymbol{v}_1^\top,\ldots,\boldsymbol{v}_n^\top]^\top \in \mathbb{R}^{nd} : (\boldsymbol{v}_j - \boldsymbol{v}_i) \in \text{null}(\boldsymbol{A}_{ij}) \forall(i,j) \in \mathcal{E}\}\}$.

A *path* is positive (nonnegative) if all the edges in the path are positive definite (positive semidefinite). The graph $\mathcal{G}$ is *connected* if there exists a nonnegative path between any two vertices in $\mathcal{G}$. A *positive tree* is a graph in which any two vertices are connected by exactly one path which is positive. A *positive spanning tree* $\mathcal{T}$ of $\mathcal{G}$ is a positive tree containing all vertices in $\mathcal{V}$. When $\boldsymbol{A}_{ij}$ and $\boldsymbol{A}_{ji}$ are not necessarily equal, i.e., $\boldsymbol{A}_{ij} \neq \boldsymbol{A}_{ji}$, the graph $\mathcal{G}$ is said to have *asymmetric matrix weights*[2].

B. Problem Formulation

Consider a system of n agents in $\mathbb{R}^d, d \geq 2$, whose interaction graph $\mathcal{G}$ is undirected and connected. Each agent $i \in \mathcal{V}$ maintains a local vector $\boldsymbol{x}_i \in \mathbb{R}^d$. Let each agent i compute the relative vectors $(\boldsymbol{x}_i - \boldsymbol{x}_j)$ to its neighbors $j \in \mathcal{N}_i$, by assuming measurement capacity or by exchanging information with its neighbors. Intuitively, in order to reach a consensus, each agent i in the

[2]The symmetry/asymmetry of the matrix weights of a graph $\mathcal{G}$, which is specified by whether $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji} \forall(i,j) \in \mathcal{E}$, or not, should be distinguished from the symmetry of the positive semidefinite matrices $\boldsymbol{A}_{ij}$.

network iteratively updates its local vector $\boldsymbol{x}_i(k+1)$, at every iteration $k+1 \geq 1$, by adding to it a matrix-weighted sum of the relative vectors, i.e., $\boldsymbol{d}_i(k) := -\sum_{j\in\mathcal{N}_i} \boldsymbol{A}_{ij}(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k))$. Here, $\boldsymbol{A}_{ij} \in \mathbb{R}^{d\times d}$ is a matrix weight associated with each edge $(i,j) \in \mathcal{E}$.

In particular, each agent $i \in \mathcal{V}$ updates $\boldsymbol{x}_i(k+1)$ via

$$\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k) - \alpha_i \sum_{j\in\mathcal{N}_i} \boldsymbol{A}_{ij}(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \quad (1)$$

$\forall k \in \mathbb{Z}^+$. Here, $\alpha_i > 0$ is a sufficiently small step size, which is used to adjust the *step length* of the displacement of $\boldsymbol{x}_i(k)$ along the update direction $\boldsymbol{d}_i(k)$ to ensure the convergence of (1) [3], [4]. In this work, we consider consensus protocols with two possible types of matrix weights $\boldsymbol{A}_{ij}$.

1) Positive semidefinite and symmetric weights $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji} \geq 0 \forall (i,j) \in \mathcal{E}$ (Section III).
2) For every $i \in \mathcal{V}$, $\boldsymbol{A}_{ij} = \boldsymbol{A}_i \in \mathbb{R}^{d\times d} \forall\ j \in \mathcal{N}_i$. That is, each agent i employs the same matrix weight $\boldsymbol{A}_i$ for every relative vector $(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \forall j \in \mathcal{N}_i$. In addition, the matrix $\boldsymbol{A}_i$ is either positive definite (Section IV) or positive semidefinite (Section V) $\forall i \in \mathcal{V}$, and it is also not required that $\boldsymbol{A}_i = \boldsymbol{A}_j$ for $i, j \in \mathcal{V}, i \neq j$.

III. Consensus Under Symmetric Matrix Weights

This section considers the consensus control for the system under the matrix-weighted consensus protocol (1) under the condition (A.1). Provided that the matrix-weighted graph $\mathcal{G}$ contains a positive spanning tree and the step sizes are sufficiently small, we show that the agents achieve a consensus. Further, asymptotic convergence to the average consensus of the system under undirected switching graphs is also ensured.

A. Matrix-Weighted Consensus Law

At an iteration $k \in \mathbb{Z}^+$, each agent i updates its state vector $\boldsymbol{x}_i(k) \in \mathbb{R}^d$ via (1). Let $\boldsymbol{x}(k) := [\boldsymbol{x}_1^\top(k), \ldots, \boldsymbol{x}_n^\top(k)]^\top$ and $\boldsymbol{G} := \text{diag}\{\alpha_i^{-1}\boldsymbol{I}_d\}_{i=1}^n$. Then, (1) can be written in a more compact form

$$\boldsymbol{x}(k+1) = (\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})\boldsymbol{x}(k). \quad (2)$$

Select $\alpha_i = (||\boldsymbol{D}_i|| + \beta_i)^{-1}$, with $\beta_i > 0$ being an arbitrary small constant, for all $i \in \mathcal{V}$. Since the matrix $\boldsymbol{G}^{-1}\boldsymbol{L}$ is nonsymmetric, in order to study the stability of the system (2), we characterize the spectral property of the matrix $\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L}$ in what follows.

Lemma 2: The matrix $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$ satisfies the following properties.

1) Its eigenvalues are real and its spectral radius is $\rho(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L}) = 1$ with the corresponding eigenvectors that are $\boldsymbol{v} \in \text{null}(\boldsymbol{L})$.
2) The unity eigenvalue 1 of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$ is *semisimple*.[3] As a result, $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$ is *semiconvergent* or, equivalently, $\lim_{k\to\infty}(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})^k = (\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})^\infty$ exists.

[3]An eigenvalue is semisimple if its algebraic multiplicity and geometric multiplicity are equal.

Proof: See Appendix A. ■

We next provide an explicit expression for the limit $\lim_{k\to\infty}(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})^k$. Consider the Jordan normal form of

$$(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L}) = \boldsymbol{V}\boldsymbol{J}\boldsymbol{V}^{-1}$$

where the matrices $\boldsymbol{V} = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{dn}]$ and $\boldsymbol{V}^{-1} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{dn}]^\top$ contain the right and left eigenvectors of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$, respectively, in which the eigenvectors corresponding to the unity eigenvalues appear earlier. Let $\boldsymbol{J} = \text{diag}(1, \ldots, 1, \boldsymbol{J}_{l_2}, \ldots, \boldsymbol{J}_{l_p}) \in \mathbb{R}^{nd\times nd}$ with the Jordan block $\boldsymbol{J}_{l_i} \in \mathbb{R}^{l_i\times l_i}, i = 2, \ldots, p, \sum_{i=1}^p l_i = dn$, corresponding to eigenvalues, whose magnitudes are less than 1. Now, we have

$$\begin{aligned}(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})^\infty &= \boldsymbol{V}\boldsymbol{J}^\infty\boldsymbol{V}^{-1} \\ &= \boldsymbol{V}\,\text{diag}(1, \ldots, 1, \boldsymbol{J}_{l_2}^\infty, \ldots, \boldsymbol{J}_{l_p}^\infty)\boldsymbol{V}^{-1} \\ &= \boldsymbol{V}\,\text{diag}(1, \ldots, 1, 0, \ldots, 0)\boldsymbol{V}^{-1} \\ &= \sum_{i=1}^{l_1} \boldsymbol{v}_i\boldsymbol{u}_i^\top \end{aligned} \quad (3)$$

where $l_1, d \leq l_1 < dn$, is the number of unity eigenvalues of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$, which is equal to the number of zero eigenvalues of $\boldsymbol{L}$ according to Lemma 2 1). In addition, the first d right eigenvectors are given as $[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d] = \boldsymbol{1}_n \otimes \boldsymbol{I}_d$ (Lemma 1). Moreover, from (3), we have $\lim_{k\to\infty} \boldsymbol{x}(k) \in \text{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{l_1})$. Thus, the following theorem is obtained whose proof is given in Appendix B.

Theorem 1: The sequence $\{\boldsymbol{x}(k)\}$ generated by (2), for an arbitrary initial vector $\boldsymbol{x}(0) \in \mathbb{R}^{dn}$, converges geometrically to $\boldsymbol{x}^* = \boldsymbol{1}_n \otimes \hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{x}} = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d]^\top\boldsymbol{x}(0) \in \mathbb{R}^d$ if and only if $\text{null}(\boldsymbol{L}) = \text{range}(\boldsymbol{1}_n \otimes \boldsymbol{I}_d)$.

Remark 1: Note that though in (1) the matrix weights are symmetric $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}$, the agents employ different step sizes α_i. Thus, the agents are shown to achieve a consensus, but not necessarily the average consensus $\bar{\boldsymbol{x}} := \frac{1}{n}(\boldsymbol{1}_n^\top \otimes \boldsymbol{I}_d)\boldsymbol{x}(0)$. In addition, the existence of a positive spanning tree in $\mathcal{G}$ is sufficient for the Laplacian matrix $\boldsymbol{L}$ to satisfy the condition in Theorem 1 [7]. However, achieving the average consensus is crucial in various problems including distributed computing and distributed data fusion in wireless sensor networks. We thus present matrix-weighted average consensus as follows.

B. Matrix-Weighted Average Consensus

Suppose that the agents use a common step size $\alpha_i = \alpha = 1/(\max_{i\in\mathcal{V}}(||\boldsymbol{D}_i||) + \beta)$ with $\beta > 0$ being an arbitrary constant, for all $i \in \mathcal{V}$. Such a step size can be computed in a distributed manner using the max-consensus algorithm [2]. As a result, the iteration (2) is rewritten as

$$\boldsymbol{x}(k+1) = (\boldsymbol{I}_{dn} - \alpha\boldsymbol{L})\boldsymbol{x}(k). \quad (4)$$

It can be shown similarly as in Lemma 2 that $(\boldsymbol{I}_{dn} - \alpha\boldsymbol{L})$ has the spectral radius of one and is semiconvergent. In addition, the columns of $(\boldsymbol{1}_n^\top \otimes \boldsymbol{I}_d)$ are the left eigenvectors corresponding to the unity eigenvalues of $(\boldsymbol{I}_{dn} - \alpha\boldsymbol{L})$, i.e., $(\boldsymbol{1}_n^\top \otimes \boldsymbol{I}_d)(\boldsymbol{I}_{dn} - \alpha\boldsymbol{L}) = (\boldsymbol{1}_n^\top \otimes \boldsymbol{I}_d)$. Therefore, we can obtain the following theorem which can be proved by following similar lines as in the Proof of Theorem 1.

Theorem 2: The sequence $\{\boldsymbol{x}(k)\}$ generated by (4), for an arbitrary initial vector $\boldsymbol{x}(0) \in \mathbb{R}^{dn}$, converges geometrically to the average consensus $\boldsymbol{x}^* = \mathbf{1}_n \otimes \bar{\boldsymbol{x}}$ if and only if $\mathrm{null}(\boldsymbol{L}) = \mathrm{range}(\mathbf{1}_n \otimes \boldsymbol{I}_d)$.

C. Matrix-Weighted Average Consensus Under Switching Network Topology

The assumption on fixed interaction graphs can be relaxed by instead considering undirected switching graphs [2], [19], [20], which impose a mild assumption on the interaction graphs and can reduce communication data significantly in each iteration (see Remark 2 below). Let $\sigma : \mathbb{Z}^+ \to \mathcal{P} := \{1, 2, \ldots, \rho\}$ be a piecewise constant switching signal. That is, there exists a subsequence $k_l, l \in \mathbb{Z}^+$, of $\{k\}_{k\in\mathbb{Z}^+}$, such that $\sigma(k)$ is a constant for $k_l \leq k < k_{l+1}, \forall k_l$.

Given a switching signal $\sigma(k)$, we define an undirected switching graph $\mathcal{G}_{\sigma(k)} = \{\mathcal{V}, \mathcal{E}_{\sigma(k)}, \mathcal{A}_{\sigma(k)}\}$, where $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E}_{\sigma(k)} := \{(i,j) \in \mathcal{V} \times \mathcal{V} : \boldsymbol{A}_{ij}(k) = \boldsymbol{A}_{ji}(k) \geq 0\}$. Note importantly that the condition $\boldsymbol{A}_{ij}(k) = \boldsymbol{A}_{ji}(k) \forall k \in \mathbb{Z}^+$, indicates that $\mathcal{G}_{\sigma(k)}$ remains undirected for every time instant k, but not necessarily connected. Let $\boldsymbol{L}_{\sigma(k)} \in \mathbb{R}^{dn\times dn}$ be the corresponding matrix-weighted Laplacian of the graph $\mathcal{G}_{\sigma(k)}$. The union of such graphs $(\mathcal{G}_{\sigma(\gamma)}, \mathcal{G}_{\sigma(\gamma+1)}, \ldots, \mathcal{G}_{\sigma(\eta)})$ over a time interval $[\gamma, \eta] \subseteq [0, \infty)$, denoted as $\mathcal{G}_{\sigma(\gamma:\eta)} := \cup_{k=\gamma}^{\eta} \mathcal{G}_{\sigma(k)}$, is defined by the triplet $\{\mathcal{V}, \mathcal{E}_{\sigma(\gamma:\eta)}, \mathcal{A}_{\sigma(\gamma:\eta)}\}$. Here, the edge set $\mathcal{E}_{\sigma(\gamma:\eta)} := \cup_{k=\gamma}^{\eta} \mathcal{E}_{\sigma(k)}$ and

$$\mathcal{A}_{\sigma(\gamma:\eta)} := \left\{ \boldsymbol{A}_{ij}(\gamma:\eta) = \sum_{k=\gamma}^{\eta} \boldsymbol{A}_{ij}(k) : (i,j) \in \mathcal{E}_{\sigma(\gamma:\eta)} \right\}.$$

It is noted that $\boldsymbol{A}_{ij}(\gamma : \eta)$ can be positive definite even if none of the weights $\{\boldsymbol{A}_{ij}(k)\}_{k\in[\gamma,\eta]}$ are positive definite. The graph $\mathcal{G}_{\sigma(k)}$ is assumed to satisfy the following joint connectedness assumption for matrix-weighted graphs [19].

Assumption 1 (Joint Connectedness): There exists a subsequence $\{k_t : t \in \mathbb{Z}^+\}$, such that $\lim_{t\to\infty} k_t = \infty$ and $k_{t+1} - k_t$ is uniformly bounded for all $t \geq 0$, and the graph $\cup_{k=k_t}^{k_{t+1}-1} \mathcal{G}_{\sigma(k)}$ contains a positive spanning tree.

Joint connectedness of switching matrix-weighted graphs implies that the union of the switching graphs over each successive finite time span $[k_t, k_{t+1} - 1]$ contains a positive spanning tree. Thus, the matrix-weighted Laplacian $\sum_{k=k_t}^{k_{t+1}-1} \boldsymbol{L}_{\sigma(k)}$ of the graph $\cup_{k=k_t}^{k_{t+1}-1} \mathcal{G}_{\sigma(k)}$ is positive semidefinite, has d zero eigenvalues, and its null space is range $(\mathbf{1}_n \otimes \boldsymbol{I}_d)$.

Consensus Law: The consensus law for each agent $i \in \mathcal{V}$ under the switching graph $\mathcal{G}_{\sigma(k)}$ is given as

$$\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k) - \alpha \sum_{j=1}^{n} \boldsymbol{A}_{ij}(k)(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \tag{5}$$

where α is a constant step size to be defined, which is common to the agents. The preceding consensus protocol can be written in a compact form

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) - \alpha \boldsymbol{L}_{\sigma(k)} \boldsymbol{x}(k). \tag{6}$$

Let $\mu := \max_{\sigma(k)} ||\boldsymbol{L}_{\sigma(k)}||$. Then, we obtain the following theorem, whose proof is given in Appendix A7-C.

Theorem 3: Suppose that Assumption 1 holds and the step size α satisfies $0 < \alpha < 1/\mu$. Then, the sequence $\{\boldsymbol{x}(k)\}$ generated by (6), for an arbitrary initial vector $\boldsymbol{x}(0) \in \mathbb{R}^{dn}$, asymptotically converges to the average consensus $\boldsymbol{x}^* = \mathbf{1}_n \otimes \bar{\boldsymbol{x}}$ as $k \to \infty$.

Theorem 3 indicates that joint connectedness condition on the switching graphs $\mathcal{G}_{\sigma(k)}$ is sufficient for the agents to achieve the average consensus, provided that the step size is sufficiently small. Further, the proof of Theorem 3 suggests that an alternative algebraic condition is $\mathrm{rank}(\sum_{k=k_t}^{k_{t+1}-1} \boldsymbol{L}_{\sigma(k)}) = dn - d$.

Remark 2: Consider the consensus of multiagent systems, whose state vectors are embedded in a high dimension. Interagent communications are then expensive if each agent i sends the whole coordinates of $\boldsymbol{x}_i(k)$ to its neighbors at every iteration. We interpret here as to how the matrix-weighted consensus law (5) would reduce the amount of exchanged data by a suitable selection of the edge weights. Although the matrix weight $\boldsymbol{A}_{ij}(k_t : k_{t+1} - 1) = \sum_{k=k_t}^{k_{t+1}-1} \boldsymbol{A}_{ij}(k)$ associated with an edge $(i,j) \in \mathcal{E}_{\sigma(k_t:k_{t+1}-1)}$ over a time interval $[k_t, k_{t+1} - 1]$ needs to be positive definite (Assumption 1), $\boldsymbol{A}_{ij}(k)$ can be simply a (relatively) low-rank positive semidefinite matrix $\forall k \in \mathbb{Z}^+$. For example, when $\boldsymbol{A}_{ij}(k) = \mathrm{diag}(\mathbf{0}, \boldsymbol{B}_k, \mathbf{0}) \in \mathbb{R}^{d\times d}$, for a matrix $\boldsymbol{B}_k \in \mathbb{R}^{r\times r}, \boldsymbol{B}_k > 0, r < d$, only r components of $\boldsymbol{A}_{ij}(k)\boldsymbol{x}_i(k)$ need to be transmitted to agent j since the other components are zeros. As a result, for each $k \in \mathbb{Z}^+$, only a small portion of the coordinates of $\boldsymbol{x}_i(k)$ is sent to j, and vice versa. Assume that $\boldsymbol{A}_{ij}(k_t : k_{t+1} - 1) > 0$, then all the coordinates of $\boldsymbol{x}_i$ evolve through interagent communications within each successive finite time span $k \in [k_t, k_{t+1} - 1], t \in \mathbb{Z}^+$. Thus, the low-rank block-diagonal matrix weight $\boldsymbol{A}_{ij}(k)$ acts as a compression operator that compresses a high-dimensional vector $\boldsymbol{x}_i(k)$ before sending it at every iteration k.

IV. Consensus Under Asymmetric Matrix Weights

This section assumes that each agent i employs the same matrix-weight $\boldsymbol{A}_i$ for every relative vector $(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \forall j \in \mathcal{N}_i$ [see condition (A.2)]. The matrix weight $\boldsymbol{A}_i$ is assumed to satisfy Assumption 2 below, for all $i \in \mathcal{V}$. Under the connectedness condition on the graph $\mathcal{G}$ and sufficiently small step sizes, we show that the system admits a consensus.

A. Consensus Law

Each agent $i \in \mathcal{V}$ updates its state vector via

$$\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k) - \alpha_i \sum_{j\in\mathcal{N}_i} \boldsymbol{A}_i(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \tag{7}$$

where $\alpha_i > 0$ is a step size associated with agent i, which is chosen sufficiently small to guarantee the convergence of (7). The matrix-weight $\boldsymbol{A}_i \in \mathbb{R}^{d\times d}$ associated with agent $i \forall i \in \mathcal{V}$ is an invertible matrix and satisfies the following condition.

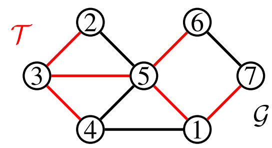


Fig. 1. Positive spanning tree $\mathcal{T}$ (in red) of $\mathcal{G}$ (– positive edges; – positive semidefinite edges).

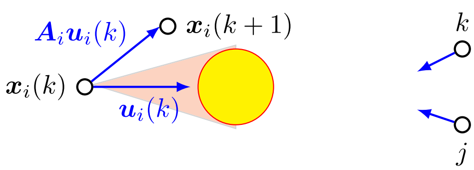


Fig. 2. Interpretation of the matrix-weighted consensus scheme (7). The desired displacement of consensus update $\boldsymbol{u}_i(k)$ and the scaled/rotated update $\boldsymbol{A}_i\boldsymbol{u}_i(k)$ of agent i.

Assumption 2: There exists a positive constant $\gamma_i > 0$, such that for any nonzero vector $\boldsymbol{y} \in \mathbb{R}^d$, the following inequality holds:

$$\boldsymbol{y}^\top \boldsymbol{A}_i^{-1}\boldsymbol{y} \geq \gamma_i||\boldsymbol{y}||^2. \tag{8}$$

Two possible classes of matrix weights that satisfy Assumption 2 are given as follows.

1) Positive definite matrix-weight $\boldsymbol{A}_i > 0$. Then, (8) is satisfied with $\gamma_i = \lambda_{\max}^{-1}(\boldsymbol{A}_i)$.
2) Rotation matrices $\boldsymbol{A}_i = \boldsymbol{R}_i \in SO(d)$ that are (nonsymmetric) positive definite, where $SO(d)$ denotes the *special orthogonal group*. In the axis-angle representation of a positive definite rotation matrix, the rotation angle is in $(-\pi/2, \pi/2)$ rad. Indeed, using the relation $\boldsymbol{R}_i^{-1} = \boldsymbol{R}_i^\top$, for every nonzero vector $\boldsymbol{y} \in \mathbb{R}^d$, we have

$$\boldsymbol{y}^\top \boldsymbol{R}_i^{-1}\boldsymbol{y} = \boldsymbol{y}^\top \boldsymbol{R}_i^\top \boldsymbol{y} = \boldsymbol{y}^\top \boldsymbol{R}_i\boldsymbol{y} > 0.$$

In addition, it follows from $\boldsymbol{y}^\top(\boldsymbol{R}_i\boldsymbol{y}) = \cos(\theta_i)||\boldsymbol{y}||^2 > 0 \Leftrightarrow \cos(\theta_i) \geq \gamma_i > 0$, where θ_i is the angle between $\boldsymbol{R}_i\boldsymbol{y}$ and $\boldsymbol{y}$, for a constant $\gamma_i \in (0,1)$. Consequently, $\boldsymbol{y}^\top \boldsymbol{R}_i^{-1}\boldsymbol{y} \geq \gamma_i||\boldsymbol{y}||^2$, which shows (8).

Remark 3: The intuition of the consensus law (7) is as follows. Let $\boldsymbol{u}_i(k) := -\sum_{j\in\mathcal{N}_i}(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k))$ be the steepest descent update direction of each agent i that minimizes the objective function $V(\boldsymbol{x}) = (1/2)\boldsymbol{x}^\top\boldsymbol{L}^o\boldsymbol{x} = 1/2\sum_{(i,j)\in\mathcal{E}}(\boldsymbol{x}_i - \boldsymbol{x}_j)^2$. Then, $\boldsymbol{A}_i\boldsymbol{u}_i(k)$ is the matrix-weighted consensus update of agent i in (7) due to the scaled matrix/rotation $\boldsymbol{A}_i$, as illustrated in Fig. 2. Furthermore, the condition

$$\begin{aligned}
&(\boldsymbol{A}_i\boldsymbol{u}_i)^\top\boldsymbol{u}_i = \\
&\alpha_i^{-2}(\boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k))^\top\boldsymbol{A}_i^{-1}(\boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k)) \\
&\overset{(8)}{\geq} \alpha_i^{-2}\gamma_i||\boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k)||^2 \geq 0
\end{aligned} \tag{9}$$

indicates that $(\boldsymbol{A}_i\boldsymbol{u}_i)$ is indeed a descent direction. Consequently, we show in Lemma 3 below that $V(\boldsymbol{x})$ is nonincreasing with respect to (7).

Remark 4: The second case 2) above also corresponds to the consensus of multiple agents in $\mathbb{R}^d$ in which the agent orientation matrices are measured with bias errors, if each agent is thought to maintain a body-fixed coordinate frame, whose origin is at its centroid, with regard to which agent measures relative vectors. Futhermore, in the case 2), the consensus law (7) is a discrete-time counterpart of the continuous-time consensus law in [9]. As a development of [9], the matrix-weighted consensus law (7) uses more general matrix weights and is applicable for an arbitrary d-dimensional space.

B. Convergence Analysis

Let $\boldsymbol{G} = \mathrm{diag}(\alpha_1\boldsymbol{A}_1, \ldots, \alpha_n\boldsymbol{A}_n)$, $\boldsymbol{x}(k) = [\boldsymbol{x}_1^\top(k), \ldots, \boldsymbol{x}_n^\top(k)]^\top$. Then, (7) can be written as

$$\boldsymbol{x}(k+1) = \boldsymbol{x}(k) - \boldsymbol{G}\boldsymbol{L}^o\boldsymbol{x}(k). \tag{10}$$

Consider the Lyapunov function $V(\boldsymbol{x}(k)) = (1/2)\boldsymbol{x}^\top(k)\boldsymbol{L}^o\boldsymbol{x}(k)$, which is positive definite w.r.t. the consensus space $\mathrm{span}(\mathbf{1}_n \otimes \boldsymbol{I}_d)$. It is noted that the gradient $\nabla V(\boldsymbol{x})$ is *Lipschitz continuous* with Lipschitz constant $L_V := ||\boldsymbol{L}^o||$, i.e., $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$

$$||\nabla V(\boldsymbol{x}) - \nabla V(\boldsymbol{y})|| = ||\boldsymbol{L}^o(\boldsymbol{x} - \boldsymbol{y})|| \leq ||\boldsymbol{L}^o||||\boldsymbol{x} - \boldsymbol{y}||.$$

An estimate of the upper-bound of the Laplacian spectral radius can be found in [21]. Let $\gamma_{\min} := \min_{i=1,\ldots n}\gamma_i$ and $\alpha_{\max} := \max_{i=1,\ldots,n}\alpha_i$. Then, we have that the Lyapunov function $V(\boldsymbol{x}(k))$ is nonincreasing according to the following lemma.

Lemma 3: Suppose that the graph $\mathcal{G}$ is connected and Assumption 2 holds. Let the step size $0 < \alpha_{\max} < 2\gamma_{\min}/L_V$. Then, the Lyapunov function $V(\boldsymbol{x}(k))$ is nonincreasing w.r.t. (10), i.e.,

$$V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) \leq -\frac{\gamma_{\min}}{\alpha_{\max}}||\boldsymbol{x}(k+1) - \boldsymbol{x}(k)||^2. \tag{11}$$

Proof: See Appendix D. ■

From Lemma 3, convergence to a consensus of the system is shown in the following result.

Theorem 4: Suppose that the graph $\mathcal{G}$ is connected and Assumption 2 holds. If $0 < \alpha_{\max} < 2\gamma_{\min}/L_V$, the sequence $\{\boldsymbol{x}(k)\}$ generated by (10), for an arbitrary vector $\boldsymbol{x}(0) \in \mathbb{R}^{dn}$, is bounded and converges geometrically to a consensus $(\mathbf{1}_n \otimes \boldsymbol{x}^*)$ with

$$\boldsymbol{x}^* = \left(\sum_{i=1}^n \frac{1}{\alpha_i}\boldsymbol{A}_i^{-1}\right)^{-1}\sum_{i=1}^n \frac{1}{\alpha_i}\boldsymbol{A}_i^{-1}\boldsymbol{x}_i(0).$$

Proof: See Appendix E. ■

Remark 5: The result of Theorem 4 further elaborates robustness to the biased measurements of the body-fixed coordinate frames of the agents and flexibility of the consensus protocol (7) in modifying both the direction and magnitude of the displacement $\boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k)$ of each agent i at each iteration k (see Fig. 2). Therefore, such flexible displacements can be utilized to design an obstacle avoidance scheme. For example, consider a stationary obstacle (the yellow circle), to which agent i measures the collision cone (the pink area), as described in Fig. 2. Then, if the rotation matrix $\boldsymbol{A}_i$ is adjustable (or otherwise, agent i can control a rotation matrix $\boldsymbol{Q}_i$, such that the angle between $\boldsymbol{u}_i$ and $\boldsymbol{Q}_i\boldsymbol{A}_i\boldsymbol{u}_i(k)$ is in $(-\pi/2, \pi/2)$ rad), one can steer the agent i's displacement to the edges of the collision cone to avoid possible collision with the obstacle. The matrix weights $\boldsymbol{A}_i$ may also be useful for modeling the biased views in the reasoning processes of networked individuals in an opinion dynamics on multiple interdependent topics [10], [11]. In this case, due to the differences in the individuals' mindsets, religions, beliefs, etc., they update their opinions with their own biased directions represented by $\boldsymbol{A}_i$. And, Theorem 4 gives a sufficient condition so that they can reach a consensus on their opinions.

V. Consensus Under Asymmetric and Positive-Semidefinite Matrix Weights

In this part, we consider the consensus scheme (7) under the scenario that the matrix weight $\boldsymbol{A}_i$ associated with agent i can be *positive semidefinite* $\forall i \in \mathcal{V}$. Therefore, $\boldsymbol{A}_i$ is not necessarily invertible and consequently the convergence analysis in Section IV is not straightforwardly applicable for this case.

A. Consensus Law

We reuse the consensus law (7) below. In particular, each agent i updates $\boldsymbol{x}_i(k)$, for an initial vector $\boldsymbol{x}_i(0) \in \mathbb{R}^d$, via

$$\boldsymbol{x}_i(k+1) = \boldsymbol{x}_i(k) - \alpha_i \sum_{j\in\mathcal{N}_i} \boldsymbol{A}_i(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \forall i \in \mathcal{V} \tag{12}$$

where $\alpha_i > 0$ is a step size and $\boldsymbol{A}_i \geq 0$ is a matrix weight, $\forall i \in \mathcal{V}$. We again use $\boldsymbol{x}(k) = [\boldsymbol{x}_1^\top(k), \ldots, \boldsymbol{x}_n^\top(k)]^\top \in \mathbb{R}^{dn}$ to denote the stacked vector of all state vectors. Let $\boldsymbol{v}_i(k) := -\sum_{j\in\mathcal{N}_i}(\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)) \in \mathbb{R}^d$ and hence (12) can be written as

$$\alpha_i \boldsymbol{A}_i \boldsymbol{v}_i(k) = \boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k). \tag{13}$$

In the sequel, we show that the state vector of agent i, $\boldsymbol{x}_i(k)$, is constrained in an affine space whose tangent space is spanned by the column space of $\boldsymbol{A}_i$.

B. Geometric Interpretation

Since the matrix weight $\boldsymbol{A}_i$ is positive semidefinite, we can decompose $\boldsymbol{A}_i$ as $\boldsymbol{A}_i = \boldsymbol{V}_i\Sigma_i\boldsymbol{V}_i^\top$, where $\Sigma_i := \mathrm{diag}(\lambda_{i,1}, \ldots, \lambda_{i,r_i}, 0, \ldots, 0) \in \mathbb{R}^{d\times d}$ with r_i $(1 \leq r_i \leq d)$ is being the rank of $\boldsymbol{A}_i$ and $\lambda_{i,l} > 0, l = 1, \ldots, r_i$, being the positive eigenvalues of $\boldsymbol{A}_i$, and $\boldsymbol{V}_i \in \mathbb{R}^{d\times d}$ is an orthogonal matrix. In addition, the first r_i columns of $\boldsymbol{V}_i$, i.e., $\boldsymbol{V}_{i,1:r_i} := [\boldsymbol{v}_{i,1}, \ldots, \boldsymbol{v}_{i,r_i}] \in \mathbb{R}^{d\times r_i}$ form an orthonormal basis of the range space of $\boldsymbol{A}_i$.

For each $i \in \mathcal{V}$ and an initial vector $\boldsymbol{x}_i(0) \in \mathbb{R}^d$, we construct a linear manifold (or an affine subspace) $\mathcal{X}_i \subseteq \mathbb{R}^d$, such that $\boldsymbol{x}_i(0) \in \mathcal{X}_i$ and the *tangent space* of $\mathcal{X}_i$, denoted as $\mathcal{TX}_i$, satisfies $\mathrm{span}(\mathcal{TX}_i) = \mathrm{span}\{\boldsymbol{v}_{i,1}, \ldots, \boldsymbol{v}_{i,r_i}\}$. It is noted that, given $\boldsymbol{x}_i(0) \in \mathbb{R}^d$, such a subspace $\mathcal{X}_i$ is unique for every $i \in \mathcal{V}$. Furthermore, since $\mathrm{range}(\boldsymbol{A}_i) = \mathrm{span}(\mathcal{TX}_i)$, it can be shown that $\boldsymbol{x}_i(k) \in \mathcal{X}_i$ for all time $k \in \mathbb{Z}^+$ $\forall i \in \mathcal{V}$. As a result, if the sequence $\{\boldsymbol{x}(k)\}$ generated by (12) converges to $\boldsymbol{1} \otimes \boldsymbol{x}^*$ for a point $\boldsymbol{x}^* \in \mathbb{R}^d$ as $k \to \infty$, then the condition in the following lemma holds.

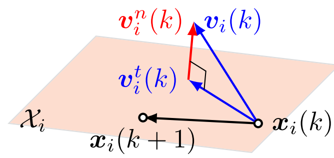


Fig. 3. Geometric illustration of Proof of Lemma 5. The tangent component $\boldsymbol{v}_i^t(k) = \boldsymbol{P}_{\mathcal{TX}_i}\boldsymbol{v}_i(k)$ and the normal component $\boldsymbol{v}_i^n(k) = (\boldsymbol{I}_d - \boldsymbol{A}_i^\dagger\boldsymbol{A}_i)\boldsymbol{v}_i(k)$. The normal vector satisfies $\boldsymbol{v}_i^n(k) \perp \Delta\boldsymbol{x}_i(k)$.

Lemma 4: A necessary condition for the agents to achieve a consensus under the iterative update (12) is the intersection of all manifolds $\mathcal{X}_i$ is nonempty $\mathcal{X} := \cap_{i=1}^n \mathcal{X}_i \neq \emptyset$.

Obviously, such a point $\boldsymbol{x}^* \in \mathcal{X}$. In addition, the intersection set $\mathcal{X}$ is either a singleton or an affine subspace.

Remark 6: The condition in Lemma 4 requires a selection of the matrix weights $\boldsymbol{A}_i$ so that $\cap_{i=1}^n \mathcal{X}_i \neq \emptyset$. Geometrically, it is required that the intersection of the degenerate subspaces $\mathcal{X}_i$ (e.g., hyperplanes and hyperlines) in the d-dimensional space is nonempty.

Define $\boldsymbol{A}_i^\dagger := \boldsymbol{V}_i\Sigma_i^\dagger\boldsymbol{V}_i^\top \in \mathbb{R}^{d\times d}$, where $\Sigma_i^\dagger := \mathrm{diag}(\lambda_{i,1}^{-1}, \ldots, \lambda_{i,r_i}^{-1}, 0, \ldots, 0) \in \mathbb{R}^{d\times d}$. Then, it can be shown that $\boldsymbol{A}_i^\dagger$ is the Moore–Penrose *generalized inverse* of $\boldsymbol{A}_i$, which satisfies: 1) $\boldsymbol{A}_i\boldsymbol{A}_i^\dagger\boldsymbol{A}_i = \boldsymbol{A}_i$, 2) $\boldsymbol{A}_i^\dagger\boldsymbol{A}_i\boldsymbol{A}_i^\dagger = \boldsymbol{A}_i^\dagger$, and 3) both $\boldsymbol{A}_i\boldsymbol{A}_i^\dagger$ and $\boldsymbol{A}_i^\dagger\boldsymbol{A}_i$ are symmetric [22]. Moreover, the *orthogonal projection matrix* that projects any vector onto the tangent space $\mathcal{TX}_i$ can be defined as

$$\boldsymbol{P}_{\mathcal{TX}_i} := \boldsymbol{A}_i^\dagger\boldsymbol{A}_i = \boldsymbol{V}_{i,1:r_i}\boldsymbol{V}_{i,1:r_i}^\top. \tag{14}$$

Note that $\boldsymbol{P}_{\mathcal{TX}_i}$ is positive semidefinite, idempotent $\boldsymbol{P}_{\mathcal{TX}_i}^2 = \boldsymbol{P}_{\mathcal{TX}_i}$, and contains r_i unity eigenvalues and the other $(d - r_i)$ eigenvalues are zeros.

C. Convergence Analysis

The following lemma is useful in showing the convergence of the system (12).

Lemma 5: Let $\Delta\boldsymbol{x}_i(k) := \boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k)$ and $\boldsymbol{v}_i(k)$ is defined above (13). Then, for all $i \in \mathcal{V}$, the following inequality holds:

$$\Delta\boldsymbol{x}_i^\top(k)\boldsymbol{v}_i(k) \geq \frac{1}{\alpha_i\lambda_{\max}}(\boldsymbol{A}_i)||\Delta\boldsymbol{x}_i(k)||^2. \tag{15}$$

Proof: First, by left-multiplying $\boldsymbol{A}_i^\dagger$ on both sides of (13), one has

$$\begin{aligned} \alpha_i\boldsymbol{A}_i^\dagger\boldsymbol{A}_i\boldsymbol{v}_i(k) &= \boldsymbol{A}_i^\dagger\Delta\boldsymbol{x}_i(k) \\ \Leftrightarrow \boldsymbol{v}_i^t(k) &= \frac{1}{\alpha_i}\boldsymbol{A}_i^\dagger\Delta\boldsymbol{x}_i(k) \end{aligned} \tag{16}$$

where $\boldsymbol{v}_i^t(k) := \boldsymbol{A}_i^\dagger\boldsymbol{A}_i\boldsymbol{v}_i(k) = \boldsymbol{P}_{\mathcal{TX}_i}\boldsymbol{v}_i(k)$ is the orthogonal projection of $\boldsymbol{v}_i(k)$ onto the tangent space $\mathcal{TX}_i$, as illustrated in Fig. 3. Let $\boldsymbol{v}_i^n(k) := \boldsymbol{v}_i(k) - \boldsymbol{v}_i^t(k) = (\boldsymbol{I}_d - \boldsymbol{A}_i^\dagger\boldsymbol{A}_i)\boldsymbol{v}_i(k)$, which is orthogonal to the tangent space $\mathcal{TX}_i$ (or normal to the linear manifold $\mathcal{X}_i$). Then, consider the inner product

$$\begin{aligned}\Delta \boldsymbol{x}_i^\top(k)\boldsymbol{v}_i(k) &= \Delta \boldsymbol{x}_i^\top(k)(\boldsymbol{v}_i^t(k)+\boldsymbol{v}_i^n(k))\\ &= \Delta \boldsymbol{x}_i^\top(k)\boldsymbol{v}_i^t(k)\\ &\overset{(16)}{=} \frac{1}{\alpha_i}\Delta \boldsymbol{x}_i^\top(k)\boldsymbol{A}_i^\dagger \Delta \boldsymbol{x}_i(k)\end{aligned}$$

where the second equality follows from the relation $\boldsymbol{v}_i^n(k) \perp \Delta\boldsymbol{x}_i(k)$ (see also Fig. 3). Moreover, it is noted that $\boldsymbol{A}_i^\dagger \geq 0$ and from (13), $\Delta\boldsymbol{x}_i(k) \perp \ker(\boldsymbol{A}_i) = \ker(\boldsymbol{A}_i^\top) = \ker(\boldsymbol{A}_i^\dagger)$, for all $k \in \mathbb{Z}^+$. As a result, it follows from the preceding equation that:

$$\begin{aligned}\Delta \boldsymbol{x}_i^\top(k)\boldsymbol{v}_i(k) &\geq \frac{1}{\alpha_i}\lambda_{\min}(\boldsymbol{A}_i^\dagger)||\Delta\boldsymbol{x}_i(k)||^2\\ &= \frac{1}{\alpha_i}\lambda_{\max}^{-1}(\boldsymbol{A}_i)||\Delta\boldsymbol{x}_i(k)||^2\end{aligned}$$

which completes the proof. ■

To proceed, we define $\boldsymbol{v}(k) := [\boldsymbol{v}_1^\top(k), \ldots, \boldsymbol{v}_n^\top(k)]^\top$ and consider the Lyapunov function

$$V(\boldsymbol{x}(k)) := (1/2)\boldsymbol{x}^\top(k)\boldsymbol{L}^\circ\boldsymbol{x}(k) = -(1/2)\boldsymbol{x}^\top(k)\boldsymbol{v}(k)$$

which is *Lipschitz differentiable* with Lipschitz constant $L_V := ||\boldsymbol{L}^\circ||$. Furthermore, let $\gamma_{\min} := \min_{i\in\mathcal{V}} \lambda_{\max}^{-1}(\boldsymbol{A}_i)$ and $\alpha_{\max} := \max_{i\in\mathcal{V}} \alpha_i$. Then, from the inequality (15) and by using a similar argument as in Proof of Lemma 3, we obtain the following result.

Lemma 6: Suppose that the graph $\mathcal{G}$ is connected. Let the step size $0 < \alpha_i < 2\gamma_{\min}/L_V, \forall i = 1, \ldots, n$. Then, the Lyapunov function $V(\boldsymbol{x}(k))$ is nonincreasing w.r.t. (12), i.e.,

$$\begin{aligned}&0 \leq V(\boldsymbol{x}(k+1)) \leq\\ &\qquad V(\boldsymbol{x}(k)) - \frac{\alpha_{\max}}{\gamma_{\min}}||\boldsymbol{x}(k+1)-\boldsymbol{x}(k)||^2. \qquad (17)\end{aligned}$$

Theorem 5: Suppose that the graph $\mathcal{G}$ is connected and for $\boldsymbol{x}(0) \in \mathbb{R}^{dn}$, the constructed linear manifolds have a nonempty intersection, $\mathcal{X} := \cap_{i=1}^n \mathcal{X}_i \neq \emptyset$. Then, if $0 < \alpha_{\max} < 2\gamma_{\min}/L_V$, the sequence $\{\boldsymbol{x}(k)\}$ generated by (12) is bounded and converges geometrically to a consensus point in $\mathcal{X}$.

Proof: See Appendix F. ■

VI. Simulation

A. Matrix-Weighted Consensus Under Switching Graphs

Consider a system of four agents whose state vectors are defined in $\mathbb{R}^3$. The graphs of the system $\mathcal{G}_\sigma, \sigma = 1,2,3,4$ are illustrated in Fig. 4(a), whose switching signal $\sigma(k), k \in \mathbb{Z}^+$ is given as follows:

$$\sigma(k_t) = \begin{cases} 1 & \text{if } k = 8t \text{ or } 8t+1\\ 2 & \text{if } k = 8t+2 \text{ or } 8t+3\\ 3 & \text{if } k = 8t+4 \text{ or } 8t+5\\ 4 & \text{if } k = 8t+6 \text{ or } 8(t+1)-1\end{cases}. \qquad (18)$$

Note that $\mathcal{G}_{\sigma(k)}$ is jointly connected in every time interval $[k_t, k_{t+1}-1] = [8t, 8t+7], t \in \mathbb{Z}^+$, while there is only one positive definite/semidefinite edge in each graph $\mathcal{G}_{\sigma(k)}, \sigma(k) \in \{1,2,3,4\}$. The matrix-weights of the system are given as

$$\boldsymbol{A}_{12}(\mathcal{G}_1) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1.2 & 0.2\\ 0 & 0.2 & 1\end{bmatrix}, \boldsymbol{A}_{14}(\mathcal{G}_2) = \begin{bmatrix}1 & 0.5 & 0\\ 0.5 & 1 & 0\\ 0 & 0 & 1.3\end{bmatrix}$$

$$\boldsymbol{A}_{23}(\mathcal{G}_3) = \begin{bmatrix}1 & 0.2 & 0\\ 0.2 & 1.2 & 0\\ 0 & 0 & 0\end{bmatrix}, \boldsymbol{A}_{23}(\mathcal{G}_4) = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0.2\\ 0 & 0.2 & 1.2\end{bmatrix}$$

and are zero matrices otherwise. It is noted that $\boldsymbol{A}_{23}(\mathcal{G}_3)$ and $\boldsymbol{A}_{23}(\mathcal{G}_4)$ are positive semidefinite, while it can be verified that $\boldsymbol{A}_{23}(\mathcal{G}_3) + \boldsymbol{A}_{23}(\mathcal{G}_4) > 0$.

The initial vectors of the agents are given as: $\boldsymbol{x}_1(0) = [-1,2,1]^\top, \boldsymbol{x}_2(0) = [1,3,2]^\top, \boldsymbol{x}_3(0) = [0,6,3]^\top$, and $\boldsymbol{x}_4(0) = [0.5,5,4]^\top$. The common step size of the agents is chosen as $\alpha = 1/3$. It can be observed in Fig. 4(b) that the agents achieve a consensus as the coordinates of $\boldsymbol{x}_i$, say x_i, y_i and $z_i, i = 1,2,3,4$, converge to the same values, respectively.

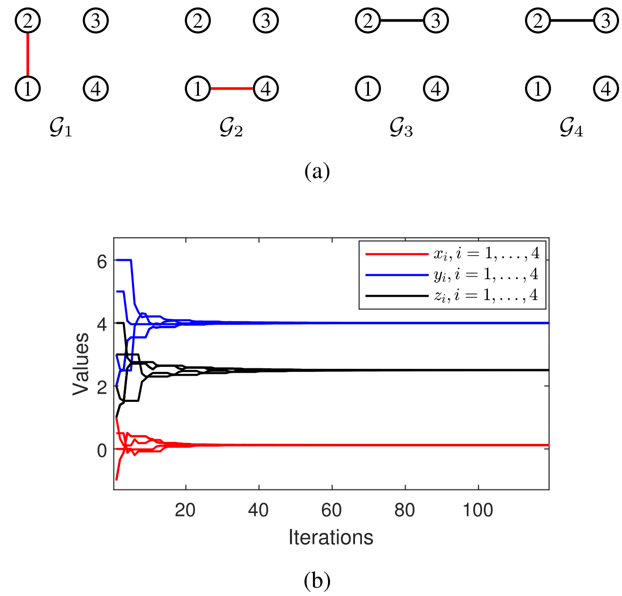
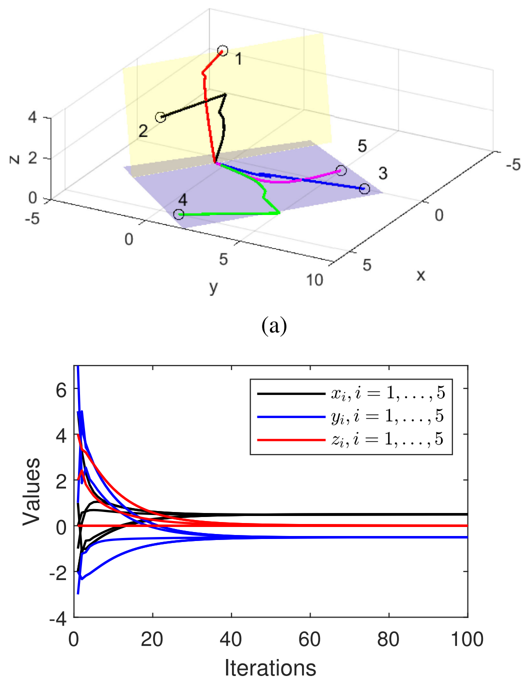


Fig. 4. Consensus of four agents under (5). (a) Switching graphs $\mathcal{G}_{\sigma(k)}$ of the network with $\mathcal{P} = \{1,2,3,4\}$ (– positive edges; – positive semidefinite edges). (b) Evolutions of the components of the agents' state vectors.

B. Consensus of Multiagent Systems With Asymmetric and Positive-Semidefinite Matrix Weights

Consider a system of five agents whose state vectors are defined in 3D and interaction graph is connected. We associate each agent i with a state vector $\boldsymbol{x}_i \in \mathbb{R}^3$. In addition, agent i can measure the relative vectors $(\boldsymbol{x}_i - \boldsymbol{x}_j)$ to neighboring agents j. The initial vectors of the agents are given as $\boldsymbol{x}_1(0) = [-2,-2,4]^\top, \boldsymbol{x}_2(0) = [1,-3,2]^\top, \boldsymbol{x}_3(0) = [0,7,0]^\top, \boldsymbol{x}_4(0) = [5,1,0]^\top$, and $\boldsymbol{x}_5(0) = [-1,5,0]^\top$. We select the step sizes of the agents as $\alpha_1 = 1/3, \alpha_2 = 2/5, \alpha_3 = 2/7$, and $\alpha_4 = \alpha_5 = 2/5$. In addition, the matrix weights of the agents are given as



(a)

(b)

Fig. 5. Consensus control of five agents in $\mathbb{R}^3$ under consensus law (12). State vectors of agents $\{1, 2\}$ and $\{3, 4, 5\}$ lie in two distinct planes. (a) Evolutions of the agents' state vectors (solid lines). (b) Evolutions of the coordinates of the agents' state vectors.

follows:

$$\boldsymbol{A}_1 = \boldsymbol{A}_2 = \begin{bmatrix} 0.6518 & -0.2604 & -0.3914 \\ -0.2604 & 0.3086 & -0.0482 \\ -0.3914 & -0.0482 & 0.4396 \end{bmatrix}$$

$$\boldsymbol{A}_3 = \boldsymbol{A}_4 = \boldsymbol{A}_5 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Such (positive semidefinite) matrix weights are chosen, such that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ lie in the plane $\mathcal{X}_1 : x + y + z = 0$, while the evolutions of $\boldsymbol{x}_3, \boldsymbol{x}_4$, and $\boldsymbol{x}_5$ are constrained in the plane $\mathcal{X}_2 : z = 0$, as illustrated in Fig. 5.

The evolutions of the agents' state vectors generated by (12) are depicted in Fig. 5. It is observed that the state vectors converge to a consensus in the set $\mathcal{X}_1 \cap \mathcal{X}_2$.

VII. Conclusion

In this article, we investigated discrete-time matrix-weighted consensus schemes for multiagent systems over undirected and connected graphs under various scenarios. When the network has symmetric matrix weights, we showed that a consensus is achieved if the agents' step sizes are sufficiently small and the interaction graph has a positive spanning tree. When the network graph is time-varying, joint connectedness condition of the network graph is sufficient for the agents to reach a consensus. In a special case of consensus with nonsymmetric matrix weights, under certain conditions, the agents are shown to a achieve a consensus.

An application of the discrete-time matrix-weighted consensus to distributed optimization and machine learning is left as future work. It will be also interesting to study discrete-time matrix-weighted consensus with asynchronous updates and over directed graphs.

Appendix

A. Proof of Lemma 2

1) We first show that $2\boldsymbol{G} - \boldsymbol{L} > 0$. Indeed, for an arbitrary nonzero vector $\boldsymbol{y} = [\boldsymbol{y}_1^\top, \ldots, \boldsymbol{y}_n^\top]^\top \in \mathbb{R}^{nd}$, we have

$$\begin{aligned} \boldsymbol{y}^\top(2\boldsymbol{G} - \boldsymbol{L})\boldsymbol{y} &= \boldsymbol{y}^\top(\boldsymbol{D} + \boldsymbol{A})\boldsymbol{y} \\ &\quad + 2\boldsymbol{y}^\top \mathrm{diag}\left(\{(||\boldsymbol{D}_i|| + \beta_i)\boldsymbol{I}_d - \boldsymbol{D}_i\}_{i=1}^n\right)\boldsymbol{y} \\ &= \sum_{(i,j)\in\mathcal{E}} (\boldsymbol{y}_i + \boldsymbol{y}_j)^\top \boldsymbol{A}_{ij}(\boldsymbol{y}_i + \boldsymbol{y}_j) \\ &\quad + 2\boldsymbol{y}^\top \mathrm{diag}\left(\{(||\boldsymbol{D}_i|| + \beta_i)\boldsymbol{I}_d - \boldsymbol{D}_i\}_{i=1}^n\right)\boldsymbol{y} > 0. \end{aligned}$$

Since $\boldsymbol{G}$ is diagonal and positive definite we can write $\boldsymbol{G} = \boldsymbol{G}^{\frac{1}{2}}\boldsymbol{G}^{\frac{1}{2}}$ with $\boldsymbol{G}^{\frac{1}{2}}$ is also a positive definite matrix. Multiplying $\boldsymbol{G}^{-\frac{1}{2}}$ on both sides of $2\boldsymbol{G} - \boldsymbol{L} > 0$ yields

$$2\boldsymbol{I}_{dn} - \boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{L}\boldsymbol{G}^{-\frac{1}{2}} > 0.$$

In addition, it is noted that $\boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{L}\boldsymbol{G}^{-\frac{1}{2}} \geq 0$ due to the positive definiteness of $\boldsymbol{G}^{-\frac{1}{2}}$ and the positive semidefiniteness of $\boldsymbol{L}$. Since the matrices $\boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{L}\boldsymbol{G}^{-\frac{1}{2}}$ and $\boldsymbol{G}^{-1}\boldsymbol{L}$ are similar, i.e., $\boldsymbol{G}^{-\frac{1}{2}}\boldsymbol{L}\boldsymbol{G}^{-\frac{1}{2}} = \boldsymbol{G}^{\frac{1}{2}}(\boldsymbol{G}^{-1}\boldsymbol{L})\boldsymbol{G}^{-\frac{1}{2}}$, they share the same spectrum. It follows that $\lambda(\boldsymbol{G}^{-1}\boldsymbol{L}) \in [0, 2)$. Consequently, $-1 < \lambda(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L}) \leq 1$ and hence $\rho(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L}) = 1$. The eigenvectors correspond to the unity eigenvalues of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$ are $\boldsymbol{v} \in \mathrm{null}(\boldsymbol{L})$.

We show 2) as follows. It follows from 1) that the eigenvectors $\boldsymbol{v}_i \in \mathbb{R}^{dn}, i = 1, \ldots l_1, d \leq l_1 < dn$ corresponding to the unity eigenvalues of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$ are the eigenvectors of $\boldsymbol{L}$ corresponding to the zero eigenvalues of $\boldsymbol{L}$. Since the Laplacian $\boldsymbol{L}$ is real symmetric, its eigenvectors are linearly independent, and so are $\{\boldsymbol{v}_i\}_{i=1}^{l_1}$. As a result, the unity eigenvalue of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$ is *semisimple* as its geometric and algebraic multiplicities are equal. This shows 2).

B. Proof of Theorem 1

It follows from (2) and (3) we have that:

$$\begin{aligned} \lim_{k\to\infty} \boldsymbol{x}(k) &= (\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})^\infty \boldsymbol{x}(0) \\ &= (\mathbf{1}_n \otimes \boldsymbol{I}_d)[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d]^\top \boldsymbol{x}(0) + \sum_{i=d+1}^{l_1} \left(\boldsymbol{u}_i^\top \boldsymbol{x}(0)\right)\boldsymbol{v}_i \\ &= (\mathbf{1}_n \otimes \boldsymbol{I}_d)\hat{\boldsymbol{x}} + \sum_{i=d+1}^{l_1} \left(\boldsymbol{u}_i^\top \boldsymbol{x}(0)\right)\boldsymbol{v}_i \end{aligned}$$

where $\hat{\boldsymbol{x}} := [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d]^\top \boldsymbol{x}(0) \in \mathbb{R}^d$. It is noted that $\boldsymbol{v}_i \perp \text{range}(\mathbf{1}_n \otimes \boldsymbol{I}_d) \forall i = d+1, \ldots, l_1$ due to the linear independence of the eigenvectors of $\boldsymbol{L}$. Moreover, such an initial vector $\boldsymbol{x}(0) \perp \text{range}\{\boldsymbol{u}_i\}_{d+1}^{l_1}$ is contained in a zero measure set. It then follows from the preceding relation that $\lim_{k\to\infty} \boldsymbol{x}(k) \to (\mathbf{1}_n \otimes \boldsymbol{I}_d)\hat{\boldsymbol{x}}$, for an arbitrary initial vector $\boldsymbol{x}(0) \in \mathbb{R}^{dn}$, if and only if $\text{null}(\boldsymbol{L}) = \mathbf{1}_n \otimes \boldsymbol{I}_d$.

We show the geometric convergence of $\boldsymbol{x}(k)$ to $(\mathbf{1}_n \otimes \hat{\boldsymbol{x}})$ as follows:

$$\begin{aligned}
&||\boldsymbol{x}(k) - (\mathbf{1}_n \otimes \hat{\boldsymbol{x}})|| = \\
&= || \left((\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})^k - (\mathbf{1}_n \otimes \boldsymbol{I}_d)[\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d] \right) \boldsymbol{x}(0)|| \\
&= ||(\boldsymbol{V}\boldsymbol{J}^k\boldsymbol{V}^{-1} - \boldsymbol{V}\boldsymbol{J}^\infty\boldsymbol{V}^{-1})\boldsymbol{x}(0)|| \\
&\leq ||\text{diag}(\mathbf{0}, \boldsymbol{J}_{l_2}^k, \ldots, \boldsymbol{J}_{l_p}^k)|| ||\boldsymbol{x}(0)|| \\
&\leq |\lambda_{d+1}|^k ||\boldsymbol{x}(0)||
\end{aligned}$$

where $|\lambda_{d+1}| < 1$ is the second largest eigenvalue in magnitude of $(\boldsymbol{I}_{dn} - \boldsymbol{G}^{-1}\boldsymbol{L})$. This completes the proof.

C. Proof of Theorem 3

It is first noted that $(\mathbf{1}_n^\top \otimes \boldsymbol{I}_d)\boldsymbol{x}(k+1) = (\mathbf{1}_n^\top \otimes \boldsymbol{I}_d)\boldsymbol{x}(0)$ is invariant with respect to (6) and so is the network centroid $\bar{\boldsymbol{x}} = (\mathbf{1}_n^\top/n \otimes \boldsymbol{I}_d)\boldsymbol{x}(k)$. Let $\tilde{\boldsymbol{x}}_i(k) = \boldsymbol{x}_i(k) - \bar{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}(k) = [\tilde{\boldsymbol{x}}_1^\top(k), \ldots, \tilde{\boldsymbol{x}}_n^\top(k)]^\top$. Then, we can rewrite (6) as

$$\tilde{\boldsymbol{x}}(k+1) = \tilde{\boldsymbol{x}}(k) - \alpha \boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k). \tag{19}$$

Consider the Lyapunov function $V(\tilde{\boldsymbol{x}}(k)) = \tilde{\boldsymbol{x}}(k)^\top\tilde{\boldsymbol{x}}(k)$, which is positive definite and radially unbounded. Then, with respect to (6) one has

$$\begin{aligned}
&V(\tilde{\boldsymbol{x}}(k+1)) - V(\tilde{\boldsymbol{x}}(k)) \\
&= \tilde{\boldsymbol{x}}(k)^\top(\boldsymbol{I}_{dn} - \alpha\boldsymbol{L}_{\sigma(k)})^\top(\boldsymbol{I}_{dn} - \alpha\boldsymbol{L}_{\sigma(k)})\tilde{\boldsymbol{x}}(k) - \tilde{\boldsymbol{x}}(k)^\top\tilde{\boldsymbol{x}}(k) \\
&= -\alpha\tilde{\boldsymbol{x}}(k)^\top(2\boldsymbol{L}_{\sigma(k)} - \alpha\boldsymbol{L}_{\sigma(k)}^2)\tilde{\boldsymbol{x}}(k) \\
&\leq -(\mu^{-1} - \alpha)\tilde{\boldsymbol{x}}(k)^\top\boldsymbol{L}_{\sigma(k)}^2\tilde{\boldsymbol{x}}(k) \\
&= -(\mu^{-1} - \alpha)||\boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k)||^2 \leq 0
\end{aligned} \tag{20}$$

where the first inequality follows from the fact that $\boldsymbol{L}_{\sigma(k)} - (1/\mu)\boldsymbol{L}_{\sigma(k)}^2 \geq 0$ with $\mu = \max_{\sigma(k)} ||\boldsymbol{L}_{\sigma(k)}||$, and in the last inequality we have used the condition $\alpha < 1/\mu$. It follows that $V(\tilde{\boldsymbol{x}}(k+1))$ is nonincreasing w.r.t. (6) and hence $\{\boldsymbol{x}(k)\}$ is bounded. In addition, $\lim_{k\to\infty} V(\tilde{\boldsymbol{x}}(k)) = \sum_{i=1}^k (V(\tilde{\boldsymbol{x}}(i)) - V(\tilde{\boldsymbol{x}}(i-1))) + V(\tilde{\boldsymbol{x}}(0))$ exists. This further implies that the sequence $\{V(\tilde{\boldsymbol{x}}(k+1)) - V(\tilde{\boldsymbol{x}}(k))\}$ is *summable* and consequently, $\lim_{k\to\infty} V(\tilde{\boldsymbol{x}}(k+1)) - V(\tilde{\boldsymbol{x}}(k)) = 0$. Thus, by (20), we have

$$\lim_{k\to\infty} \boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k) = \mathbf{0}. \tag{21}$$

Using the preceding relation, we next show that the following relation holds for all $s \in \mathbb{Z}^+$:

$$\lim_{k\to\infty} \boldsymbol{L}_{\sigma(k+s)}\tilde{\boldsymbol{x}}(k) = \mathbf{0}. \tag{22}$$

To proceed, using the relation $\tilde{\boldsymbol{x}}(k) = \tilde{\boldsymbol{x}}(k+1) + \alpha\boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k)$ [due to (19)], one has

$$\begin{aligned}
&\boldsymbol{L}_{\sigma(k+s)}\tilde{\boldsymbol{x}}(k) = \boldsymbol{L}_{\sigma(k+s)}(\tilde{\boldsymbol{x}}(k+1) + \alpha\boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k)) \\
&= \boldsymbol{L}_{\sigma(k+s)}(\tilde{\boldsymbol{x}}(k+2) + \alpha\boldsymbol{L}_{\sigma(k+1)}\tilde{\boldsymbol{x}}(k+1) + \alpha\boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k)) \\
&= \boldsymbol{L}_{\sigma(k+s)} \left(\tilde{\boldsymbol{x}}(k+s) + \alpha\boldsymbol{L}_{\sigma(k+s-1)}\tilde{\boldsymbol{x}}(k+s-1) + \cdots \right. \\
&\quad \left. + \alpha\boldsymbol{L}_{\sigma(k+1)}\tilde{\boldsymbol{x}}(k+1) + \alpha\boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k) \right).
\end{aligned}$$

Therefore, (22) follows from the fact that $\lim_{k\to\infty} \boldsymbol{L}_{\sigma(k+s)}\tilde{\boldsymbol{x}}(k+s) = \lim_{k\to\infty} \boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k) = \mathbf{0}$ for all $s \in \mathbb{Z}^+$. Moreover, it follows from (22) that:

$$\lim_{k_t\to\infty} \boldsymbol{L}_{\sigma(k_t+s)}\tilde{\boldsymbol{x}}(k_t) = \mathbf{0} \,\forall s \in \mathbb{Z}^+.$$

By summing the preceding relations over s from 0 to $(k_{t+1} - k_t - 1)$, we have

$$\lim_{k_t\to\infty} \sum_{k=k_t}^{k_{t+1}-1} \boldsymbol{L}_{\sigma(k)}\tilde{\boldsymbol{x}}(k_t) = \mathbf{0}. \tag{23}$$

Since $\tilde{\boldsymbol{x}}(k_t) \perp \text{null}(\sum_{k=k_t}^{k_{t+1}-1} \boldsymbol{L}_{\sigma(k)}) = \text{range}(\mathbf{1}_n \otimes \boldsymbol{I}_d)$ due to the joint connectedness condition in Assumption 1, we have $\lim_{k_t\to\infty} \tilde{\boldsymbol{x}}(k_t) = \mathbf{0}$. This completes the proof.

D. Proof of Lemma 3

First, it follows from (10) and (8) we have that

$$\begin{aligned}
&(\boldsymbol{x}(k+1) - \boldsymbol{x}(k))^\top\boldsymbol{L}^o\boldsymbol{x}(k) \\
&= -(\boldsymbol{x}(k+1) - \boldsymbol{x}(k))^\top\boldsymbol{G}^{-1}(\boldsymbol{x}(k+1) - \boldsymbol{x}(k)) \\
&= -\sum_{i=1}^n \frac{1}{\alpha_i}(\boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k))^\top\boldsymbol{A}_i^{-1}(\boldsymbol{x}_i(k+1) - \boldsymbol{x}_i(k)) \\
&\leq -\frac{\gamma_{\min}}{\alpha_{\max}}||\boldsymbol{x}(k+1) - \boldsymbol{x}(k)||^2.
\end{aligned}$$

Then, since ∇V is Lipschitz continuous with constant L_V, we have [23]

$$\begin{aligned}
V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) &\leq (\boldsymbol{x}(k+1) - \boldsymbol{x}(k))^\top\nabla V(\boldsymbol{x}(k)) \\
&\quad + \frac{L_V}{2}||\boldsymbol{x}(k+1) - \boldsymbol{x}(k)||^2 \\
&= (\boldsymbol{x}(k+1) - \boldsymbol{x}(k))^\top\boldsymbol{L}^o\boldsymbol{x}(k) + \frac{L_V}{2}||\boldsymbol{x}(k+1) - \boldsymbol{x}(k)||^2 \\
&\leq -\left(\frac{\gamma_{\min}}{\alpha_{\max}} - \frac{L_V}{2}\right)||\boldsymbol{x}(k+1) - \boldsymbol{x}(k)||^2 \\
&\leq 0
\end{aligned}$$

if $0 < \alpha_{\max} < 2\gamma_{\min}/L_V$.

E. Proof of Theorem 4

It follows from the nonincrease of $V(\boldsymbol{x}(k)) = \sum_{(i,j)\in\mathcal{E}} ||\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)||^2$ that $\max_{(i,j)\in\mathcal{E}} ||\boldsymbol{x}_i(k) - \boldsymbol{x}_j(k)||$

is bounded. Moreover, from (10), one has

$$\begin{aligned}&(\mathbf{1}_n \otimes \boldsymbol{I}_d)\boldsymbol{G}^{-1}\boldsymbol{x}(k+1)\\&=(\mathbf{1}_n \otimes \boldsymbol{I}_d)\boldsymbol{G}^{-1}\boldsymbol{x}(k)-(\mathbf{1}_n \otimes \boldsymbol{I}_d)\boldsymbol{G}^{-1}\boldsymbol{G}\boldsymbol{L}^{\mathrm{o}}\boldsymbol{x}(k)\\&=(\mathbf{1}_n \otimes \boldsymbol{I}_d)\boldsymbol{G}^{-1}\boldsymbol{x}(k),\end{aligned}$$

which implies that $\sum_{i=1}^{n}(\alpha_i \boldsymbol{A}_i)^{-1}\boldsymbol{x}_i(k)$ is invariant. As a result, $\{\boldsymbol{x}(k)\}$ is bounded, and hence there exist a convergent subsequence $\{\boldsymbol{x}(k_l)\}, l \in \mathbb{Z}^+$ and a limit point $\boldsymbol{x}^\dagger \in \mathbb{R}^{dn}$, such that $\lim_{l\to\infty}\boldsymbol{x}(k_l)=\boldsymbol{x}^\dagger$.

By summing up the inequalities in (11) over k from 0 to ∞, we have

$$\begin{aligned}\sum_{k=0}^{\infty}||\boldsymbol{x}(k+1)-\boldsymbol{x}(k)||^2 &\leq \frac{\alpha_{\max}}{\gamma_{\min}}\left(V(\boldsymbol{x}(0))-V(\boldsymbol{x}(\infty))\right)\\&\leq \frac{\alpha_{\max}}{\gamma_{\min}}V(\boldsymbol{x}(0)).\end{aligned}$$

It follows that $(\boldsymbol{x}(k+1)-\boldsymbol{x}(k))$ is a *square-summable* sequence and hence $||\boldsymbol{x}(k+1)-\boldsymbol{x}(k)|| \to \mathbf{0}$ as $k\to\infty$. Therefore, from (10), $||\boldsymbol{L}^{\mathrm{o}}\boldsymbol{x}(k)|| \leq ||\boldsymbol{G}^{-1}||||\boldsymbol{x}(k+1)-\boldsymbol{x}(k)|| \to 0$ as $k\to\infty$. As a result, $\boldsymbol{x}^\dagger \in \mathrm{null}(\boldsymbol{L}^{\mathrm{o}})$ and hence $\boldsymbol{x}^\dagger=\mathbf{1}_n \otimes \boldsymbol{x}^*$ for a point $\boldsymbol{x}^* \in \mathbb{R}^d$.

Since $\sum_{i=1}^{n}(\alpha_i \boldsymbol{A}_i)^{-1}\boldsymbol{x}_i(k)$ is invariant, we have

$$\begin{aligned}&\left(\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^{-1}\right)\boldsymbol{x}^*=\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^{-1}\boldsymbol{x}_i(0)\\&\Leftrightarrow \boldsymbol{x}^*=\left(\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^{-1}\right)^{-1}\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^{-1}\boldsymbol{x}_i(0),\end{aligned}$$

which is a fixed point. It follows that every sequence $\{\boldsymbol{x}(k), k \in \mathbb{Z}^+\}$ converges to $\mathbf{1}_n \otimes \boldsymbol{x}^*$.

F. Proof of Theorem 5

We first show the boundedness of the sequence $\{\boldsymbol{x}(k)\}$ generated by (12) and then prove its convergence to a consensus.

1) Boundedness Evolution: It follows from the nonincrease of $V(\boldsymbol{x}(k))=\sum_{(i,j)\in\mathcal{E}}||\boldsymbol{x}_i(k)-\boldsymbol{x}_j(k)||^2$ that $\max_{(i,j)\in\mathcal{E}}||\boldsymbol{x}_i(k)-\boldsymbol{x}_j(k)||$ is bounded. Moreover, for an arbitrary point $\boldsymbol{x}' \in \mathcal{X}$, we can rewrite (12) as

$$\begin{aligned}&(\boldsymbol{x}_i(k+1)-\boldsymbol{x}')=(\boldsymbol{x}_i(k)-\boldsymbol{x}')\\&-\alpha_i\sum_{j\in\mathcal{N}_i}\boldsymbol{A}_i\left((\boldsymbol{x}_i(k)-\boldsymbol{x}')-(\boldsymbol{x}_j(k)-\boldsymbol{x}')\right)\end{aligned}\tag{24}$$

for all $i \in \mathcal{V}$. Left-multiplying by $\alpha_i^{-1}\boldsymbol{A}_i^\dagger$ on both sides of the above equation yields

$$\begin{aligned}&\frac{1}{\alpha_i}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k+1)-\boldsymbol{x}')=\frac{1}{\alpha_i}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k)-\boldsymbol{x}')\\&-\sum_{j\in\mathcal{N}_i}\boldsymbol{P}_{\mathcal{TX}_i}\left((\boldsymbol{x}_i(k)-\boldsymbol{x}')-(\boldsymbol{x}_j(k)-\boldsymbol{x}')\right).\end{aligned}\tag{25}$$

Consider any nonzero vector $\boldsymbol{v}_i=\boldsymbol{v}_i^t+\boldsymbol{v}_i^n$, where $\boldsymbol{v}_i^t=\boldsymbol{P}_{\mathcal{TX}_i}\boldsymbol{v}_i$ and $\boldsymbol{v}_i^n=(\boldsymbol{I}_d-\boldsymbol{A}_i^\dagger\boldsymbol{A}_i)\boldsymbol{v}_i$ (see, e.g., Fig. 3). Let $\boldsymbol{P}_{\mathcal{TX}} \in \mathbb{R}^{d\times d}$ be the projection matrix that projects any vector onto the tangent space $\mathcal{TX}$. Note that when $\mathcal{X}$ is a singleton, $\boldsymbol{P}_{\mathcal{TX}}=\mathbf{0}$. Then, for every $i \in \mathcal{V}$, we have

$$\begin{aligned}&\boldsymbol{P}_{\mathcal{TX}}\boldsymbol{v}_i=\boldsymbol{P}_{\mathcal{TX}}\boldsymbol{v}_i^t\\&\Leftrightarrow \boldsymbol{P}_{\mathcal{TX}}\boldsymbol{v}_i=\boldsymbol{P}_{\mathcal{TX}}\boldsymbol{P}_{\mathcal{TX}_i}\boldsymbol{v}_i.\end{aligned}\tag{26}$$

Using the preceding relation and by left-multiplying $\boldsymbol{P}_{\mathcal{TX}}$ on both sides of (25), for all $i \in \mathcal{V}$ we obtain

$$\begin{aligned}&\frac{1}{\alpha_i}\boldsymbol{P}_{\mathcal{TX}}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k+1)-\boldsymbol{x}')=\frac{1}{\alpha_i}\boldsymbol{P}_{\mathcal{TX}}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k)-\boldsymbol{x}')\\&-\boldsymbol{P}_{\mathcal{TX}}\sum_{j\in\mathcal{N}_i}\left((\boldsymbol{x}_i(k)-\boldsymbol{x}')-(\boldsymbol{x}_j(k)-\boldsymbol{x}')\right).\end{aligned}\tag{27}$$

By adding the preceding equations over i from 1 to n, one has

$$\begin{aligned}&\boldsymbol{P}_{\mathcal{TX}}\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k+1)-\boldsymbol{x}')\\&=\boldsymbol{P}_{\mathcal{TX}}\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k)-\boldsymbol{x}')\\&\Leftrightarrow \boldsymbol{P}_{\mathcal{TX}}\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(k+1)-\boldsymbol{x}')\\&=\boldsymbol{P}_{\mathcal{TX}}\sum_{i=1}^{n}\alpha_i^{-1}\boldsymbol{A}_i^\dagger(\boldsymbol{x}_i(0)-\boldsymbol{x}'),\end{aligned}$$

which is invariant for all $k \in \mathbb{Z}^+$. Consequently, the sequence $\{\boldsymbol{x}(k)\}$ generated by (12) is bounded.

2) Convergence to a Consensus: By summing up the inequalities in (17) over k from 0 to ∞, we have

$$\begin{aligned}\sum_{k=0}^{\infty}||\boldsymbol{x}(k+1)-\boldsymbol{x}(k)||^2 &\leq \gamma_{\min}^{-1}\alpha_{\max}(V(\boldsymbol{x}(0))-V(\boldsymbol{x}(\infty)))\\&\leq \gamma_{\min}^{-1}\alpha_{\max}V(\boldsymbol{x}(0)).\end{aligned}$$

It follows that $||\boldsymbol{x}(k+1)-\boldsymbol{x}(k)|| \to \mathbf{0}$ or equivalently $\boldsymbol{x}(k) \to \hat{\boldsymbol{x}} := [\hat{\boldsymbol{x}}_1^\top,\ldots,\hat{\boldsymbol{x}}_n^\top]^\top \in \mathbb{R}^{dn}$, as $k\to\infty$. Furthermore, from (25), for an arbitrary point $\boldsymbol{x}' \in \mathcal{X}$, we have that

$$\begin{aligned}&\boldsymbol{P}_{\mathcal{TX}_i}\sum_{j\in\mathcal{N}_i}\left((\hat{\boldsymbol{x}}_i-\boldsymbol{x}')-(\hat{\boldsymbol{x}}_j-\boldsymbol{x}')\right)=\mathbf{0}\forall i \in \mathcal{V}\\&\Leftrightarrow |\mathcal{N}_i|(\hat{\boldsymbol{x}}_i-\boldsymbol{x}')=\sum_{j\in\mathcal{N}_i}\boldsymbol{P}_{\mathcal{TX}_i}(\hat{\boldsymbol{x}}_j-\boldsymbol{x}')\forall i \in \mathcal{V}\end{aligned}\tag{28}$$

where the last equality follows from $\boldsymbol{P}_{\mathcal{TX}_i}(\hat{\boldsymbol{x}}_i-\boldsymbol{x}')=(\hat{\boldsymbol{x}}_i-\boldsymbol{x}')\forall i \in \mathcal{V}$.

We define the index set $\mathcal{I} := \{i \in \mathcal{V} : i=\mathrm{argmax}_{i\in\mathcal{V}}||\hat{\boldsymbol{x}}_i-\boldsymbol{x}'||\}$. Then, consider an agent $i \in \mathcal{I}$, we have

$$\begin{aligned}\Big\|\sum_{j\in\mathcal{N}_i}\boldsymbol{P}_{\mathcal{TX}_i}(\hat{\boldsymbol{x}}_j-\boldsymbol{x}')\Big\| &\leq \sum_{j\in\mathcal{N}_i}||\boldsymbol{P}_{\mathcal{TX}_i}(\hat{\boldsymbol{x}}_j-\boldsymbol{x}')||\\&\leq \sum_{j\in\mathcal{N}_i}||\hat{\boldsymbol{x}}_j-\boldsymbol{x}'|| \leq |\mathcal{N}_i|||\hat{\boldsymbol{x}}_i-\boldsymbol{x}'||\end{aligned}$$

where the equality holds only if $\hat{\boldsymbol{x}}_j \in \mathcal{X}_i$ and $||\hat{\boldsymbol{x}}_j-\boldsymbol{x}'||=||\hat{\boldsymbol{x}}_i-\boldsymbol{x}'||$, for all $j \in \mathcal{N}_i$. This combines with (28) lead to

$\hat{\boldsymbol{x}}_j \equiv \hat{\boldsymbol{x}}_i \forall j \in \mathcal{N}_i$, and consequently, $j \in \mathcal{I} \forall j \in \mathcal{N}_i$. By repeating the above argument for all agents $j \in \mathcal{I}$ until all the agents in the system have been visited (due to the connectedness of the graph $\mathcal{G}$), we obtain $\hat{\boldsymbol{x}}_i \equiv \boldsymbol{x}^* \in \mathcal{X} \forall i \in \mathcal{V}$.

The remainder of the proof is amount to computing an explicit expression for the consensus point $\boldsymbol{x}^*$. Let $\bar{\boldsymbol{A}} := \sum_{i=1}^{n} \alpha_i^{-1} \boldsymbol{A}_i^{\dagger} \geq 0$. For an arbitrary point $\boldsymbol{x}' \in \mathcal{X}$, we have

$$\boldsymbol{P}_{\mathcal{T}\mathcal{X}} \bar{\boldsymbol{A}}(\boldsymbol{x}^* - \boldsymbol{x}') = \boldsymbol{P}_{\mathcal{T}\mathcal{X}} \sum_{i=1}^{n} \frac{1}{\alpha_i} \boldsymbol{A}_i^{\dagger}(\boldsymbol{x}_i(0) - \boldsymbol{x}')$$

$$\Leftrightarrow \boldsymbol{P}_{\mathcal{T}\mathcal{X}} \bar{\boldsymbol{A}} \boldsymbol{P}_{\mathcal{T}\mathcal{X}}(\boldsymbol{x}^* - \boldsymbol{x}') = \boldsymbol{P}_{\mathcal{T}\mathcal{X}} \sum_{i=1}^{n} \frac{1}{\alpha_i} \boldsymbol{A}_i^{\dagger}(\boldsymbol{x}_i(0) - \boldsymbol{x}')$$

where we use the relation $\boldsymbol{P}_{\mathcal{T}\mathcal{X}}(\boldsymbol{x}^* - \boldsymbol{x}') = (\boldsymbol{x}^* - \boldsymbol{x}')$. It is noted that $\mathrm{range}(\boldsymbol{P}_{\mathcal{T}\mathcal{X}}) \subseteq \mathrm{range}(\bar{\boldsymbol{A}})$ and hence $(\boldsymbol{x}^* - \boldsymbol{x}') \in \mathrm{range}(\boldsymbol{P}_{\mathcal{T}\mathcal{X}}) = \mathrm{range}(\boldsymbol{P}_{\mathcal{T}\mathcal{X}} \bar{\boldsymbol{A}} \boldsymbol{P}_{\mathcal{T}\mathcal{X}})$. Therefore, the consensus point $\boldsymbol{x}^*$ is uniquely defined as

$$\boldsymbol{x}^* = \boldsymbol{x}' + (\boldsymbol{P}_{\mathcal{T}\mathcal{X}} \bar{\boldsymbol{A}} \boldsymbol{P}_{\mathcal{T}\mathcal{X}})^{\dagger} \boldsymbol{P}_{\mathcal{T}\mathcal{X}} \sum_{i=1}^{n} \frac{1}{\alpha_i} \boldsymbol{A}_i^{\dagger}(\boldsymbol{x}_i(0) - \boldsymbol{x}')$$

where $(\boldsymbol{P}_{\mathcal{T}\mathcal{X}} \bar{\boldsymbol{A}} \boldsymbol{P}_{\mathcal{T}\mathcal{X}})^{\dagger}$ is the Moore–Penrose inverse of $\boldsymbol{P}_{\mathcal{T}\mathcal{X}} \bar{\boldsymbol{A}} \boldsymbol{P}_{\mathcal{T}\mathcal{X}}$.

References

[1] W. Ren and E. Atkins, "Distributed multi-vehicle coordinated control via local information exchange," *Int. J. Robust Nonlinear Control*, vol. 17, pp. 1002–1033, 2007.

[2] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[3] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[4] T. Yang *et al.*, "A survey of distributed optimization," *Annu. Rev. Control*, vol. 47, pp. 278–305, 2019.

[5] Q. V. Tran, Z. Sun, B. D. O. Anderson, and H.-S. Ahn, "Distributed computation of graph matching in multi-agent networks," in *Proc. 59th IEEE Conf. Decis. Control*, 2020, pp. 3139–3144.

[6] S. E. Tuna, "Synchronization under matrix-weighted laplacian," *Automatica*, vol. 73, pp. 76–81, 2016.

[7] M. H. Trinh, C. V. Nguyen, Y.-H. Lim, and H.-S. Ahn, "Matrix-weighted consensus and its applications," *Automatica*, vol. 89, pp. 415–419, 2018.

[8] L. Pan, H. Shao, M. Mesbahi, Y. Xi, and D. Li, "Bipartite consensus on matrix-valued weighted networks," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 66, no. 8, pp. 1441–1445, Aug. 2019.

[9] H.-S. Ahn and M. H. Trinh, "Consensus under biased alignment," *Automatica*, vol. 110, 2019, Art. no. 108605.

[10] N. E. Friedkin, A. V. Proskurnikov, R. Tempo, and S. E. Parsegov, "Network science on belief system dynamics under logic constraints," *Science*, vol. 354, no. 6310, pp. 321–326, 2016.

[11] H.-S. Ahn, Q. V. Tran, M. H. Trinh, M. Ye, J. Liu, and K. L. Moore, "Opinion dynamics with cross-coupling topics: Modeling and analysis," *IEEE Trans. Comput. Social Syst.*, vol. 7, no. 3, pp. 632 – 647, Mar. 2020.

[12] P. Barooah and J. P. Hespanha, "Estimation from relative measurements: Electrical analogy and large graphs," *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 2181–2193, Jun. 2008.

[13] S. Zhao and D. Zelazo, "Bearing rigidity theory and its applications for control and estimation of network systems: Life beyond distance rigidity," *IEEE Control Syst. Mag.*, vol. 39, no. 2, pp. 66–83, Apr. 2019.

[14] S. E. Tuna, "Observability through a matrix-weighted graph," *IEEE Trans. Autom. Control*, vol. 63, no. 7, pp. 2061–2074, Jul. 2018.

[15] L. Pan, H. Shao, M. Mesbahi, Y. Xi, and D. Li, "Consensus on Matrix-weighted Switching Networks," *IEEE Trans. Autom. Control*, doi: 10.1109/TAC.2021.3063115.

[16] W. Ren, R. W. Beard, and T. W. McLain, "Coordination variables and consensus building in multiple vehicle systems," in *Cooperative Control* (*Lecture Notes Control Information Science*), Berlin, Germany: Springer, vol. 309, 2004, pp. 171–188.

[17] M. H. Trinh, M. Ye, H. Ahn, and B. D. O. Anderson, "Matrix-weighted consensus with leader-following topologies," in *Proc. 11th Asian Control Conf.*, 2017, pp. 1795–1800.

[18] Y. Chinnappa and D. Constantinescu, "Average consensus in matrix-weight-balanced digraphs," in *Proc. IEEE 15th Int. Conf. Autom. Sci. Eng.*, 2019, pp. 1637–1642.

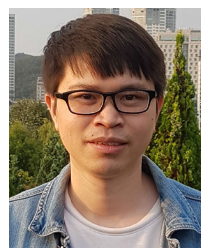
[19] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[20] Y. Su and J. Huang, "Two consensus problems for discrete-time multi-agent systems with switching network topology," *Automatica*, vol. 48, pp. 1988–1997, 2012.

[21] H. Liu, M. Lu, and F. Tian, "On the Laplacian spectral radius of a graph," *Linear Algebra Appl.*, vol. 376, pp. 135 - 141, 2004.

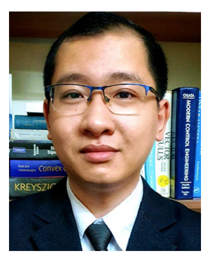
[22] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[23] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods.* Englewood Cliffs, NJ, USA: Prentice-Hall, 1989.



Quoc Van Tran received the B.E. degree in mechatronics from the Hanoi University of Science and Technology (HUST), Hanoi, Vietnam, in 2013, and the M.S. degree in mechatronics from the Gwangju Institute of Science and Technology (GIST), Gwangju, Korea, in 2015, where he is currently working toward the Ph.D. degree in mechanical engineering.

From 2016 to 2017, he was a Research Engineer with the Viettel Research and Development Institute (VRD), Viettel Group, Vietnam. His research interests include networked systems, distributed optimization and machine learning, and robotics.



Minh Hoang Trinh received the B.S. degree in electrical engineering from the Hanoi University of Science and Technology (HUST), Hanoi, Vietnam, in 2013, the M.S. degree in mechatronics and the Ph.D. degree in mechanical engineering from the Gwangju Institute of Science and Technology (GIST), Gwangju, Korea, in 2015 and 2018, respectively.

He is a Lecturer with the Department of Automatic Control, School of Electrical Engineering, HUST, Hanoi, Vietnam. In 2016, he was with Technion – Israel Institute of Technology, for a six-month research visit. Prior to joining HUST in 2019, he served as a Postdoctoral Researcher with GIST. His research interests include distributed control of multiagent systems and graph rigidity theory.

Hyo-Sung Ahn received the B.S. and M.S. degrees in astronomy from Yonsei University, Seoul, Korea, in 1998 and 2000, respectively, the M.S. degree in electrical engineering from the University of North Dakota, Grand Forks, in 2003, and the Ph.D. degree in electrical engineering from Utah State University, Logan, UT, USA, in 2006.

He is a Professor with the School of Mechanical Engineering, Gwangju Institute of Science and Technology (GIST), Gwangju, Korea. Since July 2007, he has been with the School of Mechatronics and School of Mechanical Engineering, GIST. Before joining GIST, he was a Senior Researcher with the Electronics and Telecommunications Research Institute, Daejeon, Korea. His research interests include distributed control, aerospace navigation and control, network localization, and learning control.