

Holographic entanglement density for spontaneous symmetry breaking

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ABSTRACT: We investigate the properties of the holographic entanglement entropy of the systems in which the $U(1)$ or the translational symmetry is broken *spontaneously*. For this purpose, we define the entanglement density of the strip-subsystems and examine both the first law of entanglement entropy (FLEE) and the area theorem. We classify the conditions that FLEE and/or the area theorem obey and show that such a classification may be useful for characterizing the systems. We also find universalities from both FLEE and the area theorem. In the spontaneous symmetry breaking case, FLEE is always obeyed regardless of the type of symmetry: $U(1)$ or translation. For the translational symmetry, the area theorem is always violated when the symmetry is weakly broken, independent of the symmetry breaking patterns (explicit or spontaneous). We also argue that the log contribution of the entanglement entropy from the Goldstone mode may not appear in the strongly coupled systems.

KEYWORDS: Gauge-Gravity Correspondence, Holography and Condensed Matter Physics (AdS/CMT)

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1 Introduction

Entanglement entropy has been playing an important role in the investigation of quantum gravity and quantum field theory. In particular, with the Ryu-Takayanagi formula [1, 2], the holographic duality (or gauge/gravity duality) [3–5] provides a remarkable connection between gravity theory and conformal field theory: a geometric quantity in the bulk spacetime can be related to the entanglement entropy of the boundary field theory. In other words, the Ryu-Takayanagi formula gives us a hint to understand the emergence of spacetime from the entanglement properties in the dual field theory (e.g., see also [6–10]).

In addition to its role in quantum gravity, entanglement entropy has also been used in condensed matter physics: entanglement entropy may also play an important role in characterizing and classifying the phases of matter.¹ For instance, entanglement entropy between the subsystem and the rest of the system could exhibit characteristic behavior of entanglement entropy with the subsystem size ℓ : for $(d+1)$ dimensions, entanglement entropy has a $\log(\ell^d)$ contribution from the Goldstone boson [13], $\ell^d \log \ell$ from the Fermi surface [14–17], or ℓ -independent behavior from the topologically ordered degrees of freedom [18–21].

¹See [11] for a review of quantum entanglement in condensed matter physics. See also [12] for recent progress in quantum entanglement in many-body systems using SYK model and its generalizations.

Entanglement entropy in holography. Inspired from the properties of entanglement entropy with the subsystem size ℓ in condensed matter physics, it is instructive to investigate the ℓ -dependence of entanglement entropy in holography.

Using the Ryu-Takayanagi formula (2.5), one can find interesting features of entanglement entropy in the small ($\ell \ll 1$) or large ($\ell \gg 1$) subsystem limit. For the *small* subsystem, “entanglement thermodynamics” may apply. In particular, when the system is excited entanglement entropy at $\ell \ll 1$ may obey a property analogous to the first law of thermodynamics, called *the first law of entanglement entropy*/ (FLEE) [22]:²

$$\Delta S = \frac{1}{T_{\text{ent}}} \Delta E, \quad (\ell \ll 1) \quad (1.1)$$

where $\Delta S := S - S_{\text{CFT}}$ is the increased amount of the entanglement entropy in excited states, S , compared with the ground state of the CFT, S_{CFT} (2.12). ΔE is the corresponding amount of energy in the subsystem given by

$$\Delta E = \int d^d x \langle T_{tt} \rangle, \quad (1.2)$$

where $\langle T_{tt} \rangle$ is the energy density of the excited state.³ T_{ent} in (1.1) is called the entanglement temperature [22, 35], which is proportional to the inverse of the subsystem size as $T_{\text{ent}} \sim 1/\ell$. Note that T_{ent} is universal in that it only depends on the shape of the subsystem and the dimension d , i.e., T_{ent} may not depend on the details of the excited states.

In this paper, we consider a strip subsystem of width ℓ in the x -direction in $d = 2$ (i.e., the dual gravity is asymptotically AdS_4): see figure 2. In this setup [22], ΔE in (1.2) and T_{ent} is

$$\Delta E = \ell \Omega \langle T_{tt} \rangle, \quad T_{\text{ent}} = \left(\frac{4 \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \right)^2 \frac{1}{\pi \ell}, \quad (1.3)$$

where Ω is the length in the y -direction. Thus, FLEE (1.1) implies $\Delta S \sim \langle T_{tt} \rangle \ell^2$ at small ℓ .

For the *large* subsystem, the entanglement entropy of the excited state, S , behaves as

$$S = s V + \alpha A + \dots, \quad (\ell \gg 1) \quad (1.4)$$

where s is the thermal entropy density, α is a dimensionful constant, and \dots denotes further sub-leading terms of $\mathcal{O}(\ell^{-1})$. Note that (1.4) consists of two terms: i) the “volume law” term ($\propto V = \ell \Omega$) in which V is the volume of the strip; ii) the “area law” term ($\propto A = 2 \Omega$) where A is the area of the one-dimensional boundary of the strip.⁴

The volume law term is the leading contribution of S , which may be expected for excited states, e.g., a thermal state, in that when $\ell \rightarrow \infty$ the subsystem becomes the entire system so that the minimal surface lies along the horizon [50, 51] implying that the Ryu-Takayanagi formula (2.5) may be related with the thermal entropy density s .

²FLEE has been extensively investigated in many holographic models [9, 23–48].

³Note that $\langle T_{tt} \rangle$ of the ground state of the CFT (or the pure AdS) is zero so that $\Delta \langle T_{tt} \rangle = \langle T_{tt} \rangle$.

⁴(1.4) may also contain other types of area terms such as $A \log A$ which corresponds to an area law violation [49].

The area law term [52–55] is a sub-leading contribution of S with the parameter α in which α could be expressed as a sum of several terms, in general. For instance, for the vacuum state of the CFT (in which $s = 0$), S_{CFT} in (2.12), α is a sum of two terms: one from the UV divergence $1/\epsilon$, the other $1/\ell$ [1, 2].

In addition to the appearance of thermal entropy density (s) in the leading term of S , one can also find an interesting property from the sub-leading term with α : *the area theorem* [2, 56–58]. The area theorem is a variant of the c -theorem⁵, which states that the value of α at UV/IR fixed points obeys the following inequality

$$\alpha_{\text{UV}} \geq \alpha_{\text{IR}}, \quad (1.5)$$

where the field theoretic proof was given for a sphere in $d = 3$ with strong sub-additivity in [57] or for a sphere in $d \geq 3$ with the positivity of relative entropy [58], and for a strip in $d \geq 3$ with the Null Energy Condition (NEC) [56] in which the NEC may be holographically dual to strong sub-additivity [78, 79].

Entanglement density σ . In order to examine α , it is useful to introduce the entanglement density σ [42]⁶ defined as

$$\sigma := \frac{S - S_{\text{CFT}}}{V} = \frac{\Delta S}{V}, \quad (1.6)$$

which yields a finite value because the UV divergence $1/\epsilon$ is canceled out. In terms of σ (1.6), the large subsystem behavior of the entanglement entropy (1.4) can be expressed as

$$\sigma = s - \Delta\alpha \frac{A}{V} + \dots, \quad (\ell \gg 1) \quad (1.7)$$

where the sub-leading term, A/V , is order of $\mathcal{O}(1/\ell)$ and $\Delta\alpha$ is defined

$$\Delta\alpha = \alpha_{\text{CFT}} - \alpha, \quad (1.8)$$

which may be interpreted as $\alpha_{\text{UV}} - \alpha_{\text{IR}}$ [42, 43, 82].⁷ Thus, using (1.8), the area theorem (1.5) can be rephrased as $\Delta\alpha \geq 0$.

⁵Considering how entanglement entropy behaves under a renormalization group (RG) flow, the interesting theorem, the c -theorem [59], has been derived in the two-dimensional conformal field theory, which states that the central charge c decreases along the RG flow as

$$c_{\text{UV}} \geq c_{\text{IR}},$$

where the c -function is a monotonically decreasing function of the energy scale and equals the central charge c at fixed points. For higher dimensions, there are other type of theorems without using the central charge (recall that there are more than two central charges in higher dimension). For instance, the a -theorem [60–62] with the anomaly a and the F -theorem [63, 64] with the free energy F . See also [57, 65–77].

⁶ σ is not the entanglement density [80, 81] defined as the second derivative of entanglement entropy.

⁷Recall that $s = 0$ for the entanglement entropy of the pure AdS geometry (S_{CFT}).

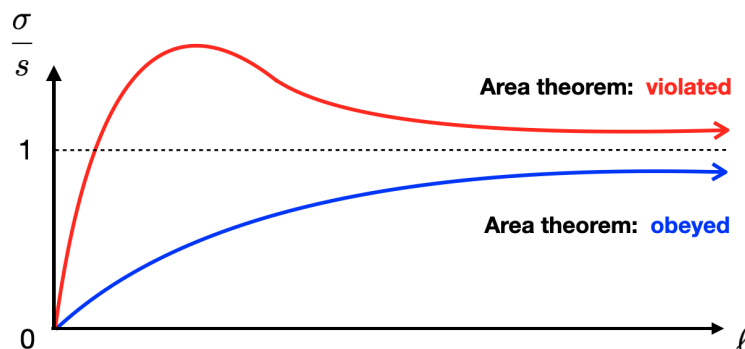


Figure 1. A schematically figure for σ in unit of s . The first law of entanglement entropy (FLEE) implies $\sigma/s \sim \ell$ at small ℓ . In the large ℓ regime, the area theorem can be obeyed when $\sigma/s \rightarrow 1^-$ (blue line) or violated when $\sigma/s \rightarrow 1^+$ (red line).

Motivations of this paper. In summary, using σ (1.6), the ℓ -dependence of holographic entanglement entropy in the small/large subsystem limit can be expressed as

$$\begin{aligned}
 (\text{The first law of entanglement entropy}) : \quad \sigma &= \langle T_{tt} \rangle T_{\text{ent}}^{-1}, & (\ell \ll 1) \\
 (\text{The area theorem}) : \quad \sigma &= s - \Delta\alpha \frac{A}{V} \quad \text{with} \quad \Delta\alpha \geq 0, & (\ell \gg 1)
 \end{aligned}
 \tag{1.9}$$

in which $T_{\text{ent}}^{-1} \sim \ell$ and $\frac{A}{V} \sim \ell^{-1}$.⁸ Note that although σ basically has the same information with ΔS because dividing by V is technically trivial, σ is a practically useful quantity in that we can easily check the area theorem $\Delta\alpha \geq 0$ by the naked eye: if $\sigma \rightarrow s^-$ at $\ell \gg 1$, this implies $\Delta\alpha \geq 0$. See figure 1.

Using (1.9), it is proposed [42] that σ may be used to characterize and classify the states of matter (or geometries in holography). In particular, depending on if FLEE and/or the area theorem is violated or not, various AdS black holes have been classified in [42]: the AdS domain wall, the hyperscaling-violating black hole, the AdS soliton, the neutral black hole, the charged black hole, the black hole with a broken translational symmetry.⁹

It is instructive to note that under which condition FLEE or the area theorem turned out to be violated in holography [42]. It was shown that the entanglement entropy in the small subsystem limit can be associated with the dimension of the perturbing operator (for instance, see the AdS-to-AdS domain walls [42]), thus FLEE may also have something to do with the dimension of the relevant couplings. For the area theorem, as its field theoretic proof [57, 58] relied on the Lorentz-invariant RG flows, one may study the violation of the area theorem by simply considering the non-zero temperature to break Lorentz invariance [42]. See also [83] for the relation between the entanglement RG monotonicity and Lorentz invariance.

In this paper, using (1.9) we apply the analysis given in [42] to the case in which symmetry is broken *spontaneously*. In particular, we consider the most well-studied sym-

⁸Note that ΔE in (1.3) is $V \langle T_{tt} \rangle$. Thus, ΔS in (1.1) is $V \langle T_{tt} \rangle T_{\text{ent}}^{-1}$ so that $\sigma = \langle T_{tt} \rangle T_{\text{ent}}^{-1}$.

⁹See also [43] for the analysis of σ with the neutral black hole, [82] for σ in a large dimension limit.

metries for the strongly correlated systems in holography [84–87]: the U(1) symmetry and the translational symmetry which can have applications to superconductors and charge density waves, respectively.

Our goal is to examine if the obedience/violation of FLEE/area theorem in (1.9) can also be used to classify the phases with *spontaneous* symmetry breaking, i.e., we extend the analysis in [42] to more realistic condensed matter phases in holography. Also, our work can be complementary to the case with a broken translational symmetry in [42] in that the translational symmetry was broken *explicitly* in [42], while it is broken *spontaneously* in our paper.

Note also that entanglement entropy for the superconducting phase (the U(1) symmetry breaking) has been extensively investigated in holography [88–115]. However, to our knowledge, our approach (FLEE/area theorem) with (1.9) has not been investigated yet for holographic superconductors.

Moreover, even for the translational symmetry breaking, entanglement entropy has only been studied with the explicit breaking case [30, 42, 116–122] and the analysis with the spontaneous breaking is still missing. Thus, in this paper, the entanglement entropy with spontaneously broken translational symmetry is analyzed for the first time.

It may also be important to make a further comment that the precise condition when the area theorem is violated ($\Delta\alpha < 0$) has not been fully understood yet. From the low temperature analysis with the various black hole geometries in [42], it is argued that the near-horizon geometry may be related to the area theorem violation: e.g., the black hole with the $\text{AdS}_2 \times R^2$ IR geometry could violate the area theorem. Thus, using spontaneous symmetry breaking, we make one step further in this direction and attempt to have a more complete understanding of the area theorem.

This paper is organized as follows. In section 2, we review entanglement entropy in holography and also introduce the useful quantity, σ , to examine (1.9). In section 3, using the holographic superconductor model, we study the obedience/violation of the FLEE/area theorem in which the U(1) symmetry is broken. In section 4, we study how the (explicitly or spontaneously) broken translational symmetry can affect both FLEE and the area theorem with the holographic axion model. Section 5 is devoted to conclusions.

2 Setup

In this section, we will review the holographic entanglement entropy in the asymptotically AdS_4 metric (2.1) and also introduce the entanglement density σ (1.6) in terms of the function of the metric. Also, we further express σ in units of the thermal entropy density s , σ/s , in order to make σ dimensionless. Such a quantity is useful not only because it is dimensionless, but also for the convenient evaluation of (1.9) in numerics (in particular for the area theorem, e.g., see figure 1).

Note that our metric (2.1) becomes the one in [42] when $h(z) = 1$. Thus, our analysis in this section corresponds to the generalization of [42] to the case with more general metrics.

2.1 Holographic entanglement density

We consider the asymptotically AdS₄ metric:

$$ds^2 = \frac{L^2}{z^2} \left[-f(z) dt^2 + \frac{dz^2}{g(z)} + h(z) (dx^2 + dy^2) \right], \quad (2.1)$$

with an AdS radius L . The functions $f(z)$, $g(z)$ and $h(z)$ in (2.1) are expanded near the AdS boundary ($z \rightarrow 0$) as

$$f(z) = 1 - \sum_{i=1} \mathbf{f}_i z^i, \quad g(z) = 1 - \sum_{i=1} \mathbf{g}_i z^i, \quad h(z) = 1 - \sum_{i=1} \mathbf{h}_i z^i, \quad (2.2)$$

where \mathbf{f}_i , \mathbf{g}_i , and \mathbf{h}_i are model dependent constants. From the holographic renormalization [123, 124], the asymptotic of the metric (2.2) determines the energy density of the dual field theory, $\langle T_{tt} \rangle$, as

$$\langle T_{tt} \rangle = \frac{L^2}{8\pi G_N} \mathbf{g}_3, \quad (2.3)$$

where G_N is Newton's constant and \mathbf{g}_3 is from (2.2). Other thermodynamic variables of the field theory, the temperature (T) and the entropy density (s), can be read with the metric (2.1) at the horizon z_h :

$$T = \frac{f'(z)}{4\pi} \sqrt{\frac{g(z)}{f(z)}} \Big|_{z_h}, \quad s = \frac{L^2}{4G_N} \frac{h(z)}{z^2} \Big|_{z_h}. \quad (2.4)$$

Holographic entanglement entropy S . One can study the entanglement entropy (S) holographically via [1, 2, 125]

$$S = \frac{\mathcal{A}_{\min}}{4G_N}, \quad (2.5)$$

where \mathcal{A}_{\min} is the area of the minimal surface in the bulk at a fixed t , which is anchored at the AdS boundary. In this paper, we consider the strip entangling surface. See figure 2: the minimal surface \mathcal{A}_{\min} is expressed as a red surface with the width of the strip ℓ in which z_* is the largest z value of the minimal surface in the bulk.

With the metric (2.1), geometric quantities for (2.5), ℓ and \mathcal{A}_{\min} , can be obtained as

$$\begin{aligned} \ell &= 2 \int_0^{z_*} dz \frac{z^2}{z_*^2 \sqrt{\frac{h(z)^2}{h(z_*)^2} - \frac{z^4}{z_*^4}}} \frac{1}{\sqrt{g(z) h(z)}}, \\ \mathcal{A}_{\min} &= 2L^2 \Omega \int_{\epsilon}^{z_*} dz \frac{1}{z^2 \sqrt{1 - \frac{z^4}{z_*^4} \frac{h(z_*)^2}{h(z)^2}}} \sqrt{\frac{h(z)}{g(z)}}, \end{aligned} \quad (2.6)$$

where the lower endpoint in \mathcal{A}_{\min} is a cutoff $z = \epsilon$ dual to the UV divergence in holography.

In order to isolate the UV divergence, it may be useful to split the integrand of \mathcal{A}_{\min} in (2.6) into two parts as

$$\int_{\epsilon}^{z_*} dz \frac{1}{z^2 \sqrt{1 - \frac{z^4}{z_*^4} \frac{h(z_*)^2}{h(z)^2}}} \sqrt{\frac{h(z)}{g(z)}} = \int_{\epsilon}^{z_*} \frac{dz}{z^2} + \int_{\epsilon \rightarrow 0}^{z_*} \frac{dz}{z^2} \left(\frac{1}{\sqrt{1 - \frac{z^4}{z_*^4} \frac{h(z_*)^2}{h(z)^2}}} \sqrt{\frac{h(z)}{g(z)}} - 1 \right), \quad (2.7)$$

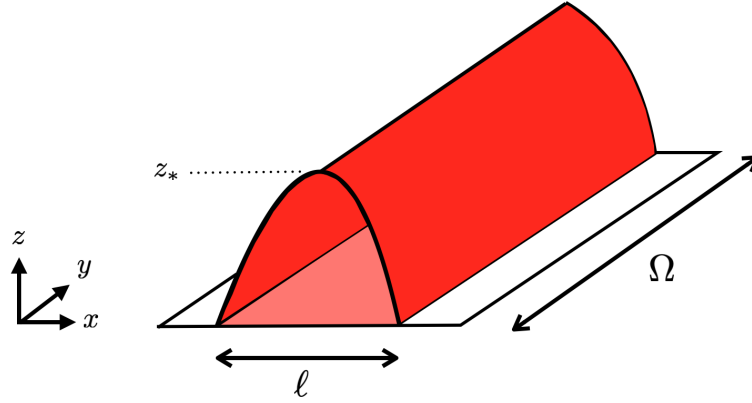


Figure 2. A strip entangling region with its minimal surface (red). The strip has width ℓ in the x -direction and length Ω in the y -direction. z_* is the largest z value of the minimal surface in the bulk.

where the UV divergence can be easily evaluated from the first term and we can let $\epsilon \rightarrow 0$ in the second integral since the second term is finite.

Using (2.7) with the change of variable $u = z/z_*$, one can express the minimal surface in (2.6) as

$$\mathcal{A}_{\min} = 2L^2\Omega \left[\frac{1}{\epsilon} - \frac{1}{z_*} + \frac{h(z_*)}{2z_*^2} \ell + \frac{1}{z_*} \int_0^1 \frac{du}{u^2} \left(\sqrt{1-u^4} \frac{h(z_*)^2}{h(z_*u)^2} \sqrt{\frac{h(z_*u)}{g(z_*u)}} - 1 \right) \right], \quad (2.8)$$

with the strip width

$$\ell = 2z_* \int_0^1 du \frac{u^2}{\sqrt{\frac{h(z_*u)^2}{h(z_*)^2} - u^4}} \frac{1}{\sqrt{g(z_*u)h(z_*u)}}. \quad (2.9)$$

Then, collecting $1/z_*$ terms in (2.8), one can find the entanglement entropy (2.5) as

$$S = \frac{L^2\Omega}{2G_N} \left[\frac{1}{\epsilon} + \frac{h(z_*)}{2z_*^2} \ell + \frac{C(z_*)}{z_*} \right], \quad (2.10)$$

with the dimensionless coefficient

$$C(z_*) := -1 + \int_0^1 \frac{du}{u^2} \left(\sqrt{1-u^4} \frac{h(z_*)^2}{h(z_*u)^2} \sqrt{\frac{h(z_*u)}{g(z_*u)}} - 1 \right). \quad (2.11)$$

Note that both (2.9) and (2.11) can be solved analytically for the case of $g(z_*u) = h(z_*u) = 1$, which produces the entanglement entropy of the pure AdS geometry, S_{CFT} , as

$$S_{\text{CFT}} = \frac{L^2\Omega}{2G_N} \left[\frac{1}{\epsilon} - \frac{2\pi}{\ell} \left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2 \right]. \quad (2.12)$$

Holographic entanglement density σ . Using (2.10)–(2.12), the entanglement density (σ) (1.6) is defined as

$$\sigma := \frac{S - S_{\text{CFT}}}{V} = \frac{L^2}{4G_N} \left[\frac{h(z_*)}{z_*^2} + \frac{C(z_*)}{z_*} \frac{2}{\ell} + \frac{4\pi}{\ell^2} \left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2 \right], \quad (2.13)$$

where the UV divergence is canceled out. V is the volume of the entangling region,¹⁰ defined as $V := \ell \Omega$, thus σ in (2.13) may specify the “density” of the entanglement entropy.

The entanglement density (2.13) is valid for any $\ell(z_*)$ via (2.9). Since the minimal surface cannot penetrate the horizon, one can notice that $\ell(z_*)$ is evaluated in the range $0 < z_* < z_h$. As explained in the introduction, the entanglement entropy can exhibit interesting features (FLEE and the area theorem) (1.9) in the small or large entangling region that can be obtained as

$$\begin{cases} z_*/z_h \rightarrow 0 : & \text{Small entangling region } (\tilde{\ell} \ll 1), \\ z_*/z_h \rightarrow 1 : & \text{Large entangling region } (\tilde{\ell} \gg 1), \end{cases} \quad (2.14)$$

where $\tilde{\ell} := \ell/z_h$ defined in (2.19).

The first law of entanglement entropy (FLEE). For $\tilde{\ell} \ll 1$, one may expand the integrand in (2.11) in the small z_*/z_h expansion and integrate it order-by-order, so that (2.13) could also be expanded in powers of $\tilde{\ell}$ via (2.9).¹¹ Then, if the geometries allow FLEE, σ will be expressed as

$$\sigma = \langle T_{tt} \rangle T_{\text{ent}}^{-1} + \dots, \quad (2.15)$$

where $\langle T_{tt} \rangle$ is (2.3), and T_{ent} (1.3). In the following section, we will explicitly perform the small z_*/z_h analysis when the black holes allow an analytic background geometry.

The area theorem. For $\tilde{\ell} \gg 1$, $z_* \rightarrow z_h$, σ in (2.13) becomes

$$\sigma = s + \frac{s z_h C(z_h)}{h(z_h)} \frac{2}{\ell} + \dots = s - \Delta\alpha \frac{A}{V} + \dots, \quad (2.16)$$

where we used s (2.4) and $A/V = 2/\ell$. We also identify

$$\Delta\alpha = -\frac{s z_h C(z_h)}{h(z_h)}, \quad (2.17)$$

which can be used for the area theorem violation: $C(z_h) > 0$ (i.e., $\Delta\alpha < 0$).¹²

¹⁰For $\text{AdS}_{d>4}$, V is the volume for higher dimensions [42, 43, 82].

¹¹We also expand the integrand in ℓ of (2.9) in small z_*/z_h and change the variable from z_*/z_h to $\tilde{\ell}$.

¹²Note that $h(z_h) \geq 0$ if $s \geq 0$ (2.4).

2.2 Dimensionless entanglement density: σ/s

Note that σ in (2.13) has a $\mathcal{O}(1/\ell^2)$ dimension. In [42, 43], in order to render σ dimensionless, s (2.4) has been used as a scaling parameter giving

$$\frac{\sigma}{s} = \frac{1}{\tilde{z}_*^2} \frac{h(\tilde{z}_*)}{h(1)} + \frac{C(\tilde{z}_*)}{\tilde{z}_* h(1)} \frac{2}{\tilde{\ell}} + \frac{4\pi}{\tilde{\ell}^2 h(1)} \left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2, \quad (2.18)$$

where tilde variables are defined as

$$\tilde{z}_* := \frac{z_*}{z_h}, \quad \tilde{\ell} := \frac{\ell}{z_h} = 2\tilde{z}_* \int_0^1 du \frac{u^2}{\sqrt{\frac{h(z_* u)^2}{h(z_*)^2} - u^4}} \frac{1}{\sqrt{g(z_* u) h(z_* u)}}, \quad (2.19)$$

with ℓ (2.9). Note that the argument for $h(z_h)$ and $C(z_*)$ have also been scaled with z_h as $h(1)$ and $C(\tilde{z}_*)$, respectively. Therefore, for the given metric (2.1), one can study σ/s via (2.18) once both $C(\tilde{z}_*)$ (2.11) and $\tilde{\ell}$ (2.19) are evaluated.

In terms of σ/s , FLEE (2.15) can be expressed as

$$\text{FLEE:} \quad \frac{\sigma}{s} = \left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} \right)^2 \frac{\tilde{\mathbf{g}}_3}{32 h(1)} \tilde{\ell} + \dots, \quad (2.20)$$

where we used (1.3) for T_{ent} , (2.3) for $\langle T_{tt} \rangle$, and (2.4) for s . We also define $\tilde{\mathbf{g}}_3 := \mathbf{g}_3 z_h^3$. One may also choose a different scaling parameter to make σ dimensionless, however, s is useful for our purpose: to check the area theorem, i.e., (2.18) at $\tilde{\ell} \gg 1$ (or $\tilde{z}_* \rightarrow 1$)

$$\text{Area theorem:} \quad \frac{\sigma}{s} = 1 + \frac{C(1)}{h(1)} \frac{2}{\tilde{\ell}} + \dots, \quad (2.21)$$

where the area theorem is violated when $C(1) > 0$.

In what follows, using holographic superconductor models and holographic axion models, we will study if FLEE (2.20) and/or the area theorem (2.21) is violated or not when the U(1) symmetry is broken in section 3, and when the translational symmetry is broken in section 4.

3 Broken U(1) symmetry

3.1 The model

We study a holographic superconductor model based on Einstein-Maxwell theory [126, 127]:¹³

$$\begin{aligned} S &= S_1 + S_2 = \int d^4x \sqrt{-g} (\mathcal{L}_1 + \mathcal{L}_2), \\ \mathcal{L}_1 &= R + 6 - \frac{1}{4} F^2, \quad \mathcal{L}_2 = -|D\Phi|^2 - M^2 |\Phi|^2, \end{aligned} \quad (3.1)$$

¹³For the recent development of holographic superconductors, we refer the reader to [128–134] and references therein.

where we set units such that the gravitational constant $16\pi G = 1$ and the AdS radius $L = 1$ for simplicity. The action (3.1) consists of two actions. S_1 is the Einstein-Maxwell theory composed of two fields: the metric $g_{\mu\nu}$, a U(1) gauge field A_μ with field strength $F = dA$. The second action S_2 is for the superconducting phase, constructed by a complex scalar field Φ with the covariant derivative $D_\mu \Phi = (\nabla_\mu - iqA_\mu) \Phi$. For numerics, we set $q = 3$ in this paper.

In order to study the action (3.1), we take the following ansatz for numerical convenience

$$\begin{aligned} ds^2 &= \frac{1}{z^2} \left[- \left(1 - \frac{z}{z_h} \right) U(z) e^{-S(z)} dt^2 + \frac{dz^2}{\left(1 - \frac{z}{z_h} \right) U(z)} + dx^2 + dy^2 \right], \\ A &= \left(1 - \frac{z}{z_h} \right) a(z) dt, \quad \Phi = z^{\Delta_-} \eta(z), \end{aligned} \quad (3.2)$$

where z_h is the horizon and the AdS boundary is at $z = 0$. Comparing (2.1) with (3.2), one can find that

$$f(z) = \left(1 - \frac{z}{z_h} \right) U(z) e^{-S(z)}, \quad g(z) = \left(1 - \frac{z}{z_h} \right) U(z), \quad h(z) = 1, \quad (3.3)$$

and the energy density ($\langle T_{tt} \rangle$), the temperature (T), and the entropy density (s) for the action (3.1) can be computed via (2.2)–(2.4).

In order for the bulk geometry asymptotic to the AdS spacetime near the boundary ($z = 0$), we impose the boundary condition for the metric as $U(0) = 1$, $S(0) = 0$. It also turns out that the boundary behavior of the matter fields A_t , Φ is

$$A_t = \mu - \rho z + \dots, \quad \Phi = \Phi^{(-)} z^{\Delta_-} + \Phi^{(+)} z^{\Delta_+} + \dots, \quad (3.4)$$

where $\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + M^2}$. According to the holographic dictionary, μ is interpreted as the chemical potential and ρ is the charge density. In the asymptotic form of Φ , $\Phi^{(-)}$ is the source and Φ^{+} is the condensate. Then, as the boundary condition for the superconducting phase, we set the source to be zero, $\Phi^{(-)} = 0$, to describe the spontaneous symmetry breaking. Thus, one can have a superconducting phase with $\Phi^{(+)} \neq 0$ and a normal phase with $\Phi = 0$. Note that, from the ansatz (3.2), one can easily read off the chemical potential μ via $\mu = a(0)$ and the source $\Phi^{(-)}$ via $\Phi^{(-)} = \eta(0)$.

3.2 Normal phase: a review

Let us first review the normal phase ($\Phi = 0$) [42], $S = S_1$ in (3.1). In the normal phase, one can find the analytic solution as

$$U(z) = 1 + \frac{z}{z_h} + \frac{z^2}{z_h^2} - \frac{\mu^2}{4} \frac{z^3}{z_h}, \quad S(z) = 0, \quad a(z) = \mu, \quad \eta(z) = 0, \quad (3.5)$$

which corresponds to

$$f(z) = g(z) = 1 - \mathbf{g}_3 z^3 + \frac{\mu^2}{4z_h^2} z^4, \quad \mathbf{g}_3 = \frac{1}{z_h^3} \left(1 + \frac{\mu^2 z_h^2}{4} \right), \quad (3.6)$$

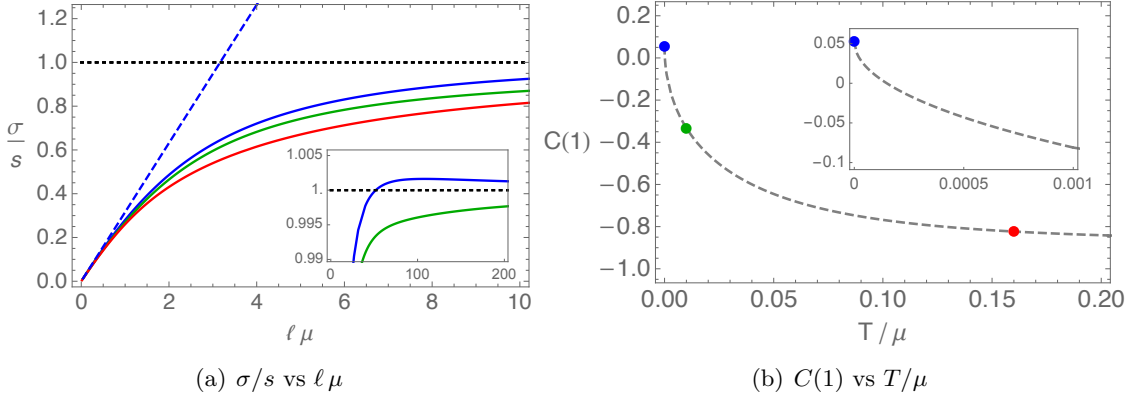


Figure 3. Entanglement density of the normal phase. *Left:* σ/s (solid lines) at $T/\mu = (0.16, 0.01, 0)$ (red, green, blue). Black dotted line is a guide line for $\sigma/s = 1$ and the blue dashed line is (3.14). The inset shows the large $\ell\mu$ behavior. *Right:* $C(1)$ (dashed line). Dots correspond to data used in figure 3(a). The inset shows the low T/μ behavior: $C(1) > 0$ at low T .

via (3.3). Note that the same notation \mathbf{g}_3 in (2.2) is used here. The temperature T (2.4) reads

$$T = \frac{3}{4\pi z_h} - \frac{\mu^2 z_h}{16\pi}. \quad (3.7)$$

The entanglement density for normal phase. In order to evaluate (2.18), we need to identify $g(z_* u)$ as

$$g(z_* u) = 1 - \tilde{\mathbf{g}}_3 \tilde{z}_*^3 u^3 + \frac{\tilde{\mu}^2}{4} \tilde{z}_*^4 u^4, \quad (3.8)$$

which is the only input function for both $C(\tilde{z}_*)$ in (2.11) and $\tilde{\ell}$ in (2.19). In (3.8), we also used $\tilde{\mathbf{g}}_3 := \mathbf{g}_3 z_h^3$, $\tilde{\mu} := \mu z_h$.

We want to fix the chemical potential for the normal phase, so σ/s should be expressed in terms of ℓ and T at fixed μ : i.e., $(\ell\mu, T/\mu)$. For this purpose, we can use

$$\ell\mu = \tilde{\ell}\tilde{\mu}, \quad \frac{T}{\mu} = \frac{1}{4\pi\tilde{\mu}} \left(3 - \frac{\tilde{\mu}^2}{4} \right), \quad (3.9)$$

where $\tilde{\ell} := \ell/z_h$ (2.19) and T from (3.7). Note that once (3.8) is being used, σ/s (2.18) can be a function of $(\tilde{z}_*, \tilde{\mu})$ which can be expressed further in terms of $(\tilde{\ell}, \tilde{\mu})$ via (2.19). Furthermore, solving the relations (3.9) we can find the expression of $(\tilde{\ell}, \tilde{\mu})$ in terms of $(\ell\mu, T/\mu)$. Thus, using (3.9), σ/s (2.18) can be evaluated at given $(\ell\mu, T/\mu)$.

At given $(\ell\mu, T/\mu)$, we investigate σ/s for the normal phase and found that

$$\text{Normal phase:} \quad \text{FLEE is obeyed, Area theorem is violated at low } T, \quad (3.10)$$

which can be seen in figure 3. In figure 3(a), we display σ/s with different $T/\mu = (0.16, 0.01, 0)$ (red, green, blue). One can find that FLEE (2.20) is obeyed for all T : $\sigma \sim \ell$ at low ℓ , e.g., see the blue dashed line ($T/\mu = 0$ case) in figure 3(a).

For the area theorem (2.21), we need to study the large ℓ behavior. From the inset in figure 3(a), one can see that the area theorem is violated at low T : e.g., $\sigma/s \rightarrow 1^+$ for the $T/\mu = 0$ case (the blue solid line). The violation of the area theorem can also be easily checked by $C(1) > 0$ at low T in figure 3(b).

FLEE of the charged black holes. We close this subsection with the analysis on how to obtain (2.20) for the charge black hole $S = S_1$ in (3.1). Our analytic result is complementary to the numerical results in [42].¹⁴

In order to study FLEE, we need to consider the $\tilde{\ell} \ll 1$ limit or equivalently $\tilde{z}_* \rightarrow 0$ limit in (3.8). First, for $\tilde{\ell}$ (2.19), one can find its leading behavior by considering $g(z_* u) = 1$ as

$$\tilde{\ell} = 2\sqrt{\pi} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \tilde{z}_* + \dots \quad (3.11)$$

Moreover, it is useful to express C (2.11) up to its sub-leading order by considering $g(z_* u) = 1 - \tilde{\mathbf{g}}_3 \tilde{z}_*^3 u^3$ in (3.8), which is

$$C(\tilde{z}_*) = -\frac{2\sqrt{\pi}}{3} \frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} + \frac{\pi}{16} \tilde{\mathbf{g}}_3 \tilde{z}_*^3 + \dots \quad (3.12)$$

Then, using (3.11), σ/s (2.18) can be expressed as

$$\frac{\sigma}{s} = \frac{1}{\tilde{z}_*^2} \left[2 + \frac{3}{\sqrt{\pi}} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} C(\tilde{z}_*) \right] + \dots, \quad (3.13)$$

where the constant 2 in (3.13) comes from the combination between the first and third terms in (2.18).¹⁵ Moreover, plugging C (3.12) into (3.13), the constant term in (3.13), 2, will be eliminated by the leading term of C in (3.12), $-\frac{2\sqrt{\pi}}{3} \frac{\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}$. Then σ/s can show FLEE (2.20) by the sub-leading term of C in (3.12) as

$$\frac{\sigma}{s} = \frac{3\sqrt{\pi}}{16} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} \tilde{\mathbf{g}}_3 \tilde{z}_* = \left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \right)^2 \frac{\tilde{\mathbf{g}}_3}{32} \tilde{\ell} = \left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \right)^2 \frac{\tilde{\mathbf{g}}_3}{32 \tilde{\mu}} (\ell \mu), \quad (3.14)$$

where we used (3.11) in the second equality and $\ell \mu = \tilde{\ell} \tilde{\mu}$ (3.9) in the last equality.

3.3 Superconducting phase

Next, let us study σ/s (2.18) of the superconducting phase ($\Phi \neq 0$), $S = S_1 + S_2$ in (3.1). Solving equations of motion from (3.1), one can find the numerical solutions $g(z_* u)$ for $T < T_c$ where T_c could be identified by the temperature at which the condensate $\Phi^{(+)}$ starts to be finite. Then, with numerical solutions, one can evaluate $C(\tilde{z}_*)$ in (2.11), $\tilde{\ell}$ in (2.19) so that σ/s in (2.18) for the superconducting phase.

¹⁴One can find the neutral case in [43].

¹⁵Recall that $h(z) = 1$ in (3.3).

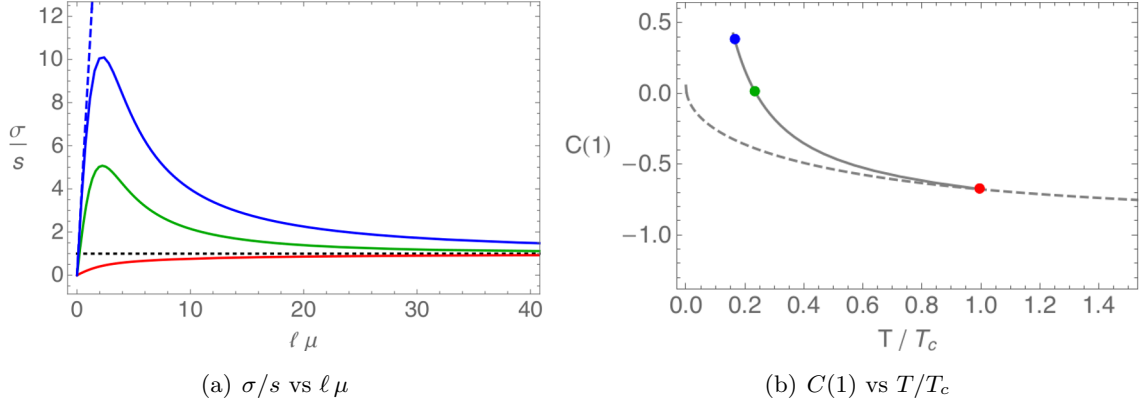


Figure 4. Entanglement density of the superconducting phase at $M^2 = 0$. (For $M^2 > 0$ we obtained qualitatively the same plots.) *Left:* σ/s (solid lines) at $T/T_c = (1, 0.23, 0.16)$ (red, green, blue). The black dotted line is the guide line for $\sigma/s = 1$ and the blue dashed line is (2.20). *Right:* $C(1)$ for the superconducting phase (solid line), the normal phase (dashed line). Dots correspond to data used in figure 4(a).

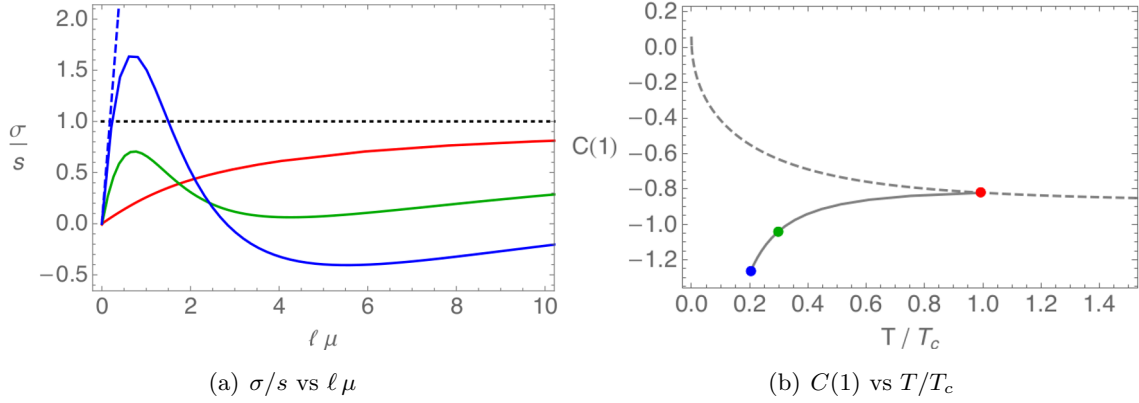


Figure 5. Entanglement density of the superconducting phase at $M^2 = -2$. *Left:* σ/s (solid lines) at $T/T_c = (1, 0.3, 0.2)$ (red, green, blue). The black dotted line is the guide line for $\sigma/s = 1$ and the blue dashed line is (2.20). *Right:* $C(1)$ for the superconducting phase (solid line), the normal phase (dashed line). Dots correspond to data used in figure 5(a).

In particular, we make the plot of σ/s for two cases: i) $M^2 = 0$ in figure 4; ii) $M^2 = -2$ in figure 5 in order to examine the mass (M^2) dependence of σ/s .¹⁶ These are the representative examples for $M^2 \geq 0$ and $M^2 < 0$, respectively.¹⁷

The entanglement density for the superconducting phase. For $M^2 \geq 0$, we found that σ/s for superconductors is qualitatively similar to the normal phase (3.10), i.e., FLEE is obeyed (see figure 4(a)), while the area theorem is violated at low T (see figure 4(b)).¹⁸

¹⁶See also [135] for the M^2 dependence of another quantum information quantity: the holographic complexity.

¹⁷We also checked the $M^2 = 2$ case. However, the result of $M^2 = 2$ is qualitatively the same as $M^2 = 0$.

¹⁸Note that (2.20) for the superconducting phase can be evaluated with $\tilde{\ell} = \ell\mu/\tilde{\mu}$ (3.9) with $\tilde{\mathbf{g}}_3$ from the numerical solution of $g(z)$ (2.2) and $\tilde{\mu}$ from $a(0)$ (3.2). Note also that we set $z_h = 1$ for numerics, thus $\mu = \tilde{\mu}$ and $\mathbf{g}_3 = \tilde{\mathbf{g}}_3$.

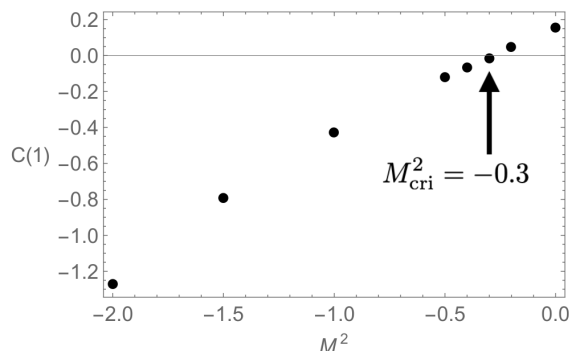


Figure 6. $C(1)$ vs M^2 at $T/T_c = 0.2$.

On the other hand, the $M^2 < 0$ case can produce the different result from the $M^2 \geq 0$ case: both FLEE and the area theorem are obeyed. For FLEE, see figure 5(a) in which $\sigma \sim \ell$ at low ℓ for all $T < T_c$. For the area theorem, see figure 5(b): $C(1)$ for superconducting phase (solid line) may always be negative, indicating the area theorem is obeyed for all $T < T_c$.

Note that $C(1)$ for the superconducting phase in figure 5(b) does not reach to $T/T_c = 0$ because of the instability in our numerics. However, from the large $\ell\mu$ behavior of σ/s in figure 5(a), it may be expected that the area theorem for the superconducting phase is obeyed even at lower T/T_c because, in the large $\ell\mu$ regime, the value of σ/s is smaller at lower T , e.g., at $\ell\mu = 10$, σ/s is decreasing from 0.8 ($T/T_c = 1$, red) to -0.2 ($T/T_c = 0.2$, blue). This implies that $C(1)$ in (2.21) is decreasing as T is lowered, which is consistent with figure 5(b). Thus, σ/s may be approaching 1^- for all $T < T_c$.

In figure 6, we also make a plot of $C(1)$ at $T/T_c = 0.2$ in order to find the critical value of M^2 , M_{cri}^2 , in which the area theorem is violated $M^2 \gtrsim M_{\text{cri}}^2$. We found $M_{\text{cri}}^2 \sim -0.3$.¹⁹ Unlike the area theorem, the mass (M^2) dependence may not be manifested in FLEE.

In summary, for holographic superconductors, the obedience/violation of FLEE/area theorem depends on M^2 :

$$\begin{aligned} \text{Superconducting phase with } M^2 \geq M_{\text{cri}}^2 : & \text{ FLEE is obeyed, Area theorem is violated at small } T, \\ \text{Superconducting phase with } M^2 < M_{\text{cri}}^2 : & \text{ FLEE is obeyed, Area theorem is obeyed.} \end{aligned} \quad (3.15)$$

If the area theorem were obeyed in the superconducting phase, the violation of the area theorem would have played the role of distinguishing the normal phase from the superconducting phase. However, this is not the case. Instead, it seems that the violation of the area theorem may classify the superconducting phases: one class with $M^2 \geq M_{\text{cri}}^2$ and the other with $M^2 < M_{\text{cri}}^2$. Noting that M^2 is related to the material properties such as the critical temperature T_c in condensed matter systems we may say M^2 can identify different superconducting materials.

For a summary of FLEE/area theorem, see table 1.

¹⁹We would like to thank the referee for pointing this out.

U(1) symmetry	FLEE	Area theorem
Normal phase	Obeded	Violated
Superconducting phase ($M^2 \geq M_{\text{cri}}^2$)	Obeded	Violated
Superconducting phase ($M^2 < M_{\text{cri}}^2$)	Obeded	Obeded

Table 1. FLEE/area theorem with broken U(1) symmetry. The area theorem is violated at low T . $M_{\text{cri}}^2 \sim -0.3$.

4 Broken translational symmetry

4.1 The model

We consider the Einstein-Maxwell-Axion model [136] as

$$S = \int d^4x \sqrt{-g} \left[R + 6 - \frac{1}{4} F^2 - X^N \right], \quad (4.1)$$

which is constructed with S_1 in (3.1) by adding an additional scalar field φ_i called the axion field

$$X := \frac{1}{2} \sum_{i=1}^2 (\partial \varphi_i)^2, \quad \varphi_i = m x^i, \quad (4.2)$$

where m denotes the strength of the translational symmetry breaking.

The action (4.1) allows analytic background solutions as

$$ds^2 = \frac{1}{z^2} \left[-f(z) dt^2 + \frac{1}{g(z)} dz^2 + d\vec{x}^2 \right], \quad A = A_t(z) dt, \quad (4.3)$$

with

$$f(z) = g(z) = 1 - \mathbf{g}_3 z^3 + \frac{\mu^2}{4z_h^2} z^4 + \frac{m^{2N}}{2(2N-3)} z^{2N}, \quad A_t(z) = \mu \left(1 - \frac{z}{z_h} \right), \quad (4.4)$$

where it becomes (3.6) at $m = 0$ and \mathbf{g}_3 is determined by $f(z_h) = 0$ as

$$\mathbf{g}_3 = \frac{1}{z_h^3} \left(1 + \frac{\mu^2 z_h^2}{4} + \frac{m^{2N} z_h^{2N}}{2(2N-3)} \right), \quad (4.5)$$

and the temperature T (2.4) is

$$T = \frac{3}{4\pi z_h} - \frac{\mu^2 z_h}{16\pi} - \frac{m^{2N} z_h^{2N-1}}{8\pi}. \quad (4.6)$$

Translational symmetry breaking in holography. The holographic model (4.1) has been used to study various translational symmetry breaking patterns: explicitly broken translational invariance (EXB), spontaneously broken translational invariance (SSB).

Solving the equations of motion near the AdS boundary ($z \rightarrow 0$), one can check that the power of the potential in (4.1), N , determines the boundary behavior of the axion field²⁰

$$\varphi_i = \varphi_i^{(0)} + \varphi_i^{(1)} z^{5-2N} + \dots, \quad (4.7)$$

where $\varphi_i^{(0)} = m x^i$ corresponds to the bulk solution (4.2). According to the holographic dictionary, the leading term can be interpreted as the source and the sub-leading term is for the vacuum expectation value (vev). Thus, when $N < 5/2$, $\varphi_i^{(0)}$ is the source so that the translational symmetry is broken explicitly, via $\varphi_i^{(0)} = m x^i$. On the other hand, when $N > 5/2$, $\varphi_i^{(0)}$ is no longer the source, instead it is the vev so one can study the spontaneously broken translational symmetry by taking $\varphi_i^{(1)} = 0$ with $N > 5/2$. In summary, using the action (4.1), one can study the broken translational symmetry in holography as

$$\begin{cases} N < 5/2 : & \text{Explicitly broken translational symmetry} & (\text{EXB}), \\ N > 5/2 : & \text{Spontaneously broken translational symmetry} & (\text{SSB}). \end{cases} \quad (4.8)$$

The entanglement density with broken translational symmetry. In what follows, we study σ/s (2.18) with various values of N . For this purpose, we identify $g(z_* u)$ from (4.4) as

$$g(z_* u) = 1 - \tilde{\mathbf{g}}_3 \tilde{z}_*^3 u^3 + \frac{\tilde{\mu}^2}{4} \tilde{z}_*^4 u^4 + \frac{\tilde{m}^{2N}}{2(2N-3)} \tilde{z}_*^{2N} u^{2N}, \quad (4.9)$$

where we used $\tilde{\mathbf{g}}_3 := \mathbf{g}_3 z_h^3$, $\tilde{\mu} := \mu z_h$, and $\tilde{m} := m z_h$. Moreover, a similar relation as in (3.9) can also be used at finite m

$$\ell \mu = \tilde{\ell} \tilde{\mu}, \quad \frac{T}{\mu} = \frac{1}{4\pi \tilde{\mu}} \left[3 - \frac{\tilde{\mu}^2}{4} - \frac{\tilde{\mu}^{2N}}{2} \left(\frac{m}{\mu} \right)^{2N} \right], \quad (4.10)$$

where T is from (4.6). Solving the relations (4.10) one can find the expression of $(\tilde{\ell}, \tilde{\mu})$ in terms of $(\ell \mu, T/\mu, m/\mu, N)$. Thus, σ/s in (2.18) can be evaluated as a function of $(\ell \mu, T/\mu)$ at given $(m/\mu, N)$.

4.2 First law of entanglement entropy

Considering the $\tilde{z}_* \rightarrow 0$ limit, let us first discuss FLEE (2.20) for general N . For easy comparison with the normal phase ($m = 0$), we follow similar analysis as given in (3.11)–(3.14).

From the leading term of (4.9), $g(z_* u) = 1$, one can find the expression for small $\tilde{\ell}$ as (3.11). Next, as in (3.12), we also need to express $C(\tilde{z}_*)$ by considering $g(z_* u)$ up to its sub-leading order. However, as can be seen from (4.9), the sub-leading term of $g(z_* u)$ depends on N as

$$\begin{cases} N < 3/2 : & g(z_* u) = 1 + \frac{\tilde{m}^{2N}}{2(2N-3)} \tilde{z}_*^{2N} u^{2N} + \dots, \\ N > 3/2 : & g(z_* u) = 1 - \tilde{\mathbf{g}}_3 \tilde{z}_*^3 u^3 + \dots, \end{cases} \quad (4.11)$$

where \dots denotes higher order terms.²¹ For the $N > 3/2$ case, we have the same sub-leading term of the normal phase ($m = 0$) so that FLEE is obeyed at $N > 3/2$ as (3.14).

²⁰See also [137] for the description of the translational symmetry breaking with more general holographic models.

²¹Note that the model (4.1) may not be well defined at $N = 3/2$, e.g., see (4.5).

The violation of FLEE at $N < 3/2$. However, for the $N < 3/2$ case, we need to compute $C(\tilde{z}_*)$ (2.11) with the different sub-leading term, $g(z_* u) = 1 + \frac{\tilde{m}^{2N}}{2(2N-3)} \tilde{z}_*^{2N} u^{2N}$, which produces

$$C(\tilde{z}_*) = -\frac{2\sqrt{\pi}}{3} \frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{5}{4})} + \frac{\sqrt{\pi}}{32(3-2N)} \frac{\Gamma(\frac{N}{2} - \frac{1}{4})}{\Gamma(\frac{N}{2} + \frac{5}{4})} \tilde{m}^{2N} \tilde{z}_*^{2N}. \quad (4.12)$$

Then, plugging (4.12) into (3.13), σ/s can be further expressed as

$$\begin{aligned} \frac{\sigma}{s} &= \frac{3}{32(3-2N)} \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} \frac{\Gamma(\frac{N}{2} - \frac{1}{4})}{\Gamma(\frac{N}{2} + \frac{5}{4})} \left(\frac{m}{\mu}\right)^{2N} \tilde{\mu}^{2N} \tilde{z}_*^{2N-2}, \\ &= \underbrace{\frac{3}{32(3-2N)} \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{7}{4})} \frac{\Gamma(\frac{N}{2} - \frac{1}{4})}{\Gamma(\frac{N}{2} + \frac{5}{4})} \left(\frac{2\sqrt{\pi}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right)^{2-2N} \left(\frac{m}{\mu}\right)^{2N} \tilde{\mu}^2 (\ell\mu)^{2N-2}}_{:=\mathcal{C}}, \end{aligned} \quad (4.13)$$

where the sub-leading correction in (4.12) only survives for the same reason explained above (3.14). In (4.13), we also used $\tilde{m} = \frac{m}{\mu} \tilde{\mu}$ in the first equality and (3.11), (4.10) in the last equality.

One can also find the higher order corrections to (4.13) by considering $g(z_* u)$ up to the $\mathcal{O}(\tilde{z}_*^3)$ order, $g(z_* u) = 1 + \frac{\tilde{m}^{2N}}{2(2N-3)} \tilde{z}_*^{2N} u^{2N} - \tilde{\mathbf{g}}_3 \tilde{z}_*^3 u^3$, which gives (3.14) as a correction, i.e.,

$$\frac{\sigma}{s} = \mathcal{C} (\ell\mu)^{2N-2} + \left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)^2 \frac{\tilde{\mathbf{g}}_3}{32\tilde{\mu}} (\ell\mu) + \dots, \quad (4.14)$$

where the leading term is (4.13). Therefore, for $N < 3/2$, one can notice that FLEE is violated due to the leading term (4.13): $\sigma/s \sim (\ell\mu)^{2N-2 < 1}$. Note that when $N = 1$ our result (4.14) reproduces [42] in which the leading term (4.13) becomes a constant \mathcal{C} .

Based on the analysis (4.11)–(4.14), we find the violation of FLEE depending on the symmetry breaking pattern (4.8) as

$$\begin{aligned} \text{EXB : } &\begin{cases} \text{FLEE is violated} & (N < 3/2), \\ \text{FLEE is obeyed} & (3/2 < N < 5/2), \end{cases} \\ \text{SSB : } &\text{FLEE is obeyed} \quad (N > 5/2). \end{aligned} \quad (4.15)$$

Note that our result (4.15) is complementary to the previous work [42] ($N = 1$ case), and shows that EXB does not guarantee the violation of FLEE, e.g., $3/2 < N < 5/2$.

In figure 7, we make the representative plots of σ/s for FLEE with different symmetry breaking patterns, $N = 1$ (EXB) in figure 7(a) and $N = 3$ (SSB) in figure 7(b), in order to show that our analytic result is consistent with the numerical result of σ/s .²²

²²Note that our analytic analysis is valid for all T .

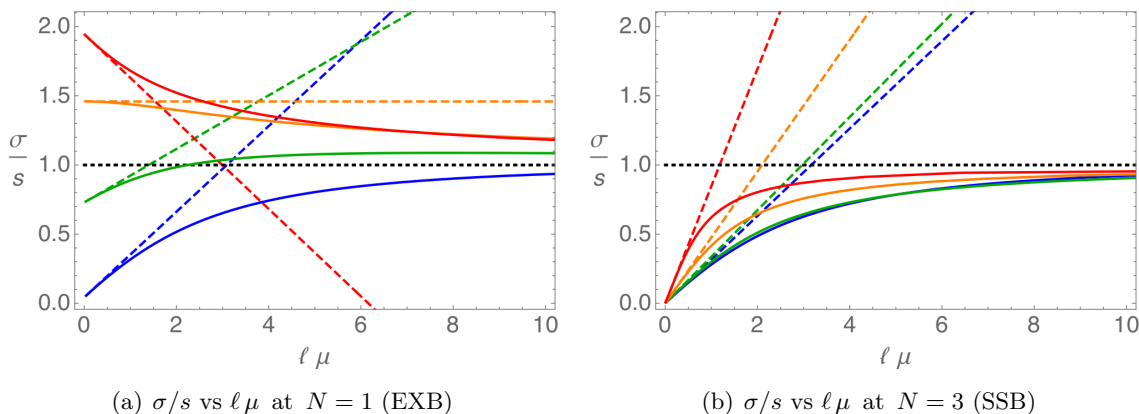


Figure 7. Entanglement density at $T/\mu = 0$, $m/\mu = (2, 1, 0.5, 0)$ (red, orange, green, blue). The black dotted line is the guide line for $\sigma/s = 1$. Solid lines are numerical results σ/s via (2.18), dashed lines are analytic results: (4.14) in figure 7(a), (3.14) in figure 7(b). Note that the $m/\mu = 0$ case corresponds to the blue data in figure 3(a).

4.3 Area theorem

Next, let us study $C(1)$ to check the area theorem (2.21). Note that, unlike the analysis for FLEE, we should resort to numerics in order for checking the area theorem [42].

In particular, we examine the area theorem by the sign of $C(1)$ at $T = 0$ as in the normal phase ($m/\mu = 0$) in figure 3(b).²³ In other words, we study how the blue dot in figure 3(b) behaves when we increase m/μ at given N . See figure 8.

In figure 8(a), we found that the translational symmetry breaking, a finite m/μ , can change the sign of $C(1)$. For instance, at $N = 3$ (the red line in figure 8(a)), one can see that the positive value (blue dot) of $C(1)$ can become negative as we increase m/μ . Here we denote the critical value of m/μ giving $C(1) = 0$ as m_c/μ : i.e., the area theorem is violated ($C(1) > 0$) at $m < m_c$.²⁴

From figure 8(a), one can also find that m_c/μ (stars) depends on the value of N , i.e., $m_c = m_c(N)$. In particular, m_c/μ tends to increase as we decrease N , e.g., from $N = 3$ (red star) to $N = 2.3$ (orange star). As we decrease the value of N further, one can find the critical N , N_{cri} , at which $C(1)$ is always non-negative (black dashed line). Here we numerically found $N_{\text{cri}} \sim 2.1$. This N -dependent behavior of m_c/μ can be seen clearly in figure 8(b).

²³In all holographic models in this paper, we checked that $C(1)$ has the monotonic behavior with respect to T (e.g., figure 3–5). We also checked that $C(1)$ at finite m/μ has a monotonic behavior for T as well, which is similar to figure 3. Thus the $T = 0$ analysis of $C(1)$ may be enough to check the violation of the area theorem.

²⁴In figure 8(a), the red/orange star represents m_c/μ .

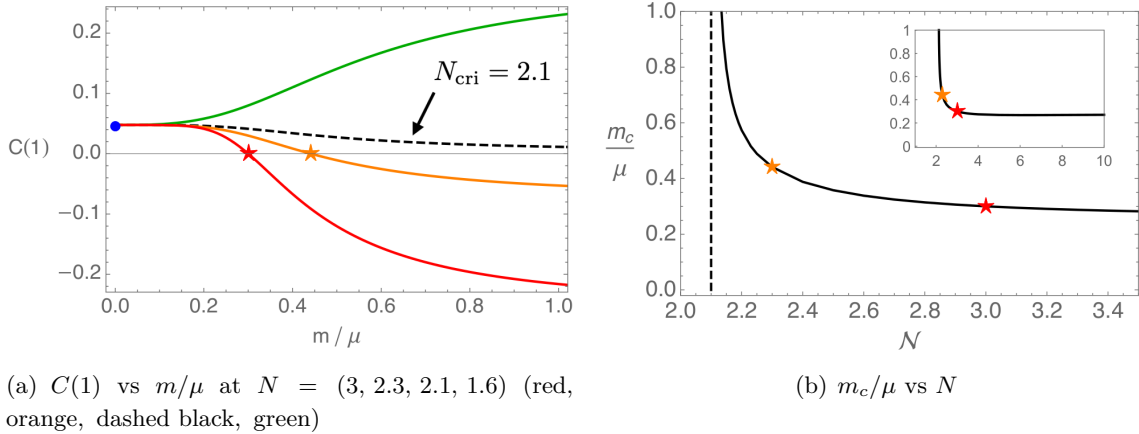


Figure 8. Checking the area theorem at $T = 0$ with m/μ . Blue dot in figure 8(a) is the same as in figure 3(b). The stars represent m_c/μ and the black dashed lines are the data at N_{cri} .

From all the figures in figure 8, we find the condition for the violation of the area theorem, $C(1) > 0$, depending on the symmetry breaking pattern (4.8) as

$$\begin{aligned}
 \text{EXB} : \quad & \begin{cases} C(1) > 0 \text{ at any } m & (N < N_{\text{cri}}), \\ C(1) > 0 \text{ at } m < m_c(N) & (N_{\text{cri}} < N < 5/2), \end{cases} \\
 \text{SSB} : \quad & \begin{cases} C(1) > 0 \text{ at } m < m_c(N) & (5/2 < N < 5), \\ C(1) > 0 \text{ at } m < m_c & (N \gtrsim 5), \end{cases}
 \end{aligned} \tag{4.16}$$

where $N_{\text{cri}} \sim 2.1$.²⁵ Note that at $N > N_{\text{cri}} \sim 2.1$, m_c is a function of N in general as can be seen in figure 8(b). However, we found that m_c can be an N -independent universal value, $m_c/\mu \sim 0.27$, at $N \gtrsim 5$, which corresponds to the last line in (4.16). See the inset of figure 8(b).

Based on (4.15) together with (4.16), we construct the summary table of the obedience/violation of FLEE/area theorem in the presence of the translational symmetry breaking. See table 2. As in the U(1) symmetry breaking in table 1, we find that the obedience/violation of FLEE/area theorem may also be used to classify the phases in which translational invariance is broken spontaneously. We also found the universality at $N \gtrsim 5$ with the N -independent m_c . Note that, like M^2 in holographic superconductors, for the SSB, N can also be closely related to the material properties such as the shear modulus, $G(N)$, [136].

Further comments on the violation of the area theorem. It is argued [42] that the violation of the area theorem at low T may be related to the IR geometry of black holes. For instance, the black hole with the $\text{AdS}_2 \times R^2$ IR geometry [42] turns out to violate the

²⁵One can easily check the first line of (4.16) in figure 8(a): at $N < N_{\text{cri}}$, i.e., from a dashed black towards a green, $C(1) > 0$ at any m/μ .

Translational symmetry	FLEE	Area theorem
Explicit breaking ($N < 3/2$)	Violated	Violated at any m
Explicit breaking ($3/2 < N < N_{\text{cri}}$)	Obeeyed	Violated at any m
Explicit breaking ($N_{\text{cri}} < N < 5/2$)	Obeeyed	Violated at $m < m_c(N)$
Spontaneous breaking ($5/2 < N < 5$)	Obeeyed	Violated at $m < m_c(N)$
Spontaneous breaking ($N \gtrsim 5$)	Obeeyed	Violated at $m < m_c$

Table 2. FLEE/area theorem with broken translation symmetry. The violation of the area theorem is examined at $T = 0$ and we find $N_{\text{cri}} \sim 2.1$.

area theorem, i.e., $C(1) > 0$. However, our result (4.16) may be one counter example for this argument because one can find that the area theorem can be obeyed ($C(1) < 0$) at $m > m_c$ for the black hole (4.1) in which its IR geometry is the $\text{AdS}_2 \times R^2$ [138].²⁶

Furthermore, our result with the superconducting case, table 1, can also be another counter example. The near-horizon geometry of the superconducting phase at zero temperature has been systematically investigated in [144, 145]. For instance, it was shown that a near-horizon geometry is the Lifshitz geometry for $M^2 > 0$, while it is the AdS_4 for $M^2 = 0$.

Thus, one can notice that the violation of the area theorem may not be connected to the IR geometry even for the superconducting phase in that we found the violation of the area theorem for both $M^2 > 0$ and $M^2 = 0$.²⁷

Entanglement entropy and Goldstone modes in holography. As mentioned in the introduction, entanglement entropy may have an $N_G \log(\ell/\epsilon)$ contribution from the Goldstone mode [13, 139, 140] where N_G is the number of Goldstone modes.²⁸ Let us close this section with a discussion for the appearance of such a log contribution in holography.

For SSB ($N > 5/2$) (4.8), it is shown that there can be (transverse/longitudinal) phonons, the Goldstone modes of the translational symmetry, in holography [136]. Thus, it will be instructive to see if SSB in holography can also show the log term in the entanglement entropy.

Note that the log term comes with the cutoff divergence, $\log \epsilon$, so that we may need to investigate the UV-divergence structure in (2.7), which can be analysed by the expansion of the integrand near the AdS boundary ($z \rightarrow 0$) as

$$\int_{\epsilon}^{z_*} dz \frac{1}{z^2 \sqrt{1 - \frac{z^4}{z_*^4} \frac{h(z_*)^2}{h(z)^2}}} \sqrt{\frac{h(z)}{g(z)}} = \int_{\epsilon}^{z_*} dz \left[\frac{1}{z^2} + \frac{h'(0) - g'(0)}{2z} + \dots \right]. \quad (4.17)$$

²⁶See the appendix in [138], showing that the action (4.1) has the $\text{AdS}_2 \times R^2$ IR geometry.

²⁷We would like to thank the referee for the valuable suggestion.

²⁸Entanglement entropy has been investigated for coplanar antiferromagnets with $\text{SO}(3)$ symmetry in [13, 139], the spin-1/2 XY model [140].

One can see that the leading term in (4.17) produces the usual UV divergence term $1/\epsilon$ and there can be an additional log divergence from the sub-leading term as $(g'(0) - h'(0)) \log \epsilon$ which is our interest.

However, for SSB ($N > 5/2$), it turns out such a log term in the holographic model (4.1) is vanishing in that $g(z)$ does not include a linear order in z (4.4) with $h(z) = 1$, so $g'(0) - h'(0) = 0$. Thus, based on this result, we speculate that the log contribution of the entanglement entropy from the Goldstone mode may not appear in the strongly correlated systems.

5 Conclusions

We have studied entanglement entropy with the *spontaneously* broken symmetry in holography. In particular, using entanglement density (1.6) in the small and large subsystem region, we examine if the obedience/violation of FLEE/area theorem (1.9) can classify the phases in which the U(1) or translational symmetry is broken.

This kind of classification has been done for explicit-symmetry-breaking cases in [42] and we extend it to spontaneous-symmetry-breaking cases. We show that indeed the obedience/violation of FLEE/area theorem may characterize the phases where the U(1) or the translational symmetry is broken spontaneously. For the summary see table 1 or table 2, where M^2 and N may be related to the physical properties of materials such as T_c or the shear modulus respectively.

Furthermore, we find some universalities from the classification. First, FLEE is always obeyed with the spontaneous symmetry breaking regardless of the type of symmetry: U(1) or translational. Second, independent of translational symmetry breaking patterns (EXB or SSB), the area theorem is always violated when the translational symmetry is weakly broken (small m/μ). Third, for SSB, in particular $N \gtrsim 5$, the area theorem may not be related to the material properties (i.e., independent of N).

As a byproduct, we find that the violation of the area theorem may not be related to the IR geometry unlike the speculation in [42]. At first we speculated that the condition for the violation of the area theorem may have something to do with the symmetry breaking but we found it was not the case. Understanding the precise condition for the violation of the area theorem remains an open problem and requires further investigation. We also argue that the log contribution of the entanglement entropy from the Goldstone mode may not appear in strongly correlated systems.

Although the obedience/violation of the FLEE/area theorem can classify phases further within the spontaneously symmetry broken phase, it may not detect the phase transition between symmetry broken/unbroken phases (e.g., the normal phase vs the superconducting phase with $M^2 \geq M_{\text{cri}}^2$) or the symmetry breaking patterns (e.g., EXB at $N_{\text{cri}} < N < 5/2$ vs SSB at $5/2 < N < 5$).

Based on this work, it will be interesting to investigate other quantum information properties, in the presence of spontaneous symmetry breaking, such as mutual information, entanglement wedge cross section, complexity [141], and the entanglement entropy of the

Hawking radiation, i.e., the Page curve.²⁹ We leave this subject as future work and will address them in the near future.

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²⁹See [[142](#), [143](#)] for a recent review of quantum information in holography.

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