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Sign rigidity theory and application to formation specification control☆

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ABSTRACT

This paper develops a sign rigidity theory to characterize and examine multi-agent rigid formations consistent with a formation specification including distance- and signed area-constraints, and to control an arbitrarily positioned set of agents to take up the specifications. The sign rigidity theory can be viewed as an extended version of the standard distance rigidity theory with the addition of signed area constraints. This property enables elimination of possible formation specification ambiguities arising when a formation specification includes distance constraints only. As an application of the sign rigidity theory, this paper explores formation specification control in 2-D space. Under the proposed gradient-based formation control law, almost global convergence (from arbitrary initial positions) can be achieved when a target formation consists of triangulated sub-formations defined by distance- and signed area-constraints; the formation control law is applied for either single-integrator models or unicycle models.

1. Introduction

Distributed formation control has been extensively studied in recent decades with a growing interest in multi-agent systems (Ahn, 2020; Oh & Ahn, 2018; Oh, Park, & Ahn, 2015; Park & Ahn, 2016), where a formation is typically viewed as a collection of point agents in 2-D or 3-D space. Formation control includes formation shape control, flocking control, maneuvering, affine control, etc. A fundamental objective of formation control is for a group of agents to maintain or achieve a target formation shape consistent with specific constraints, where the term of formation shape will be defined precisely at a later point. According to the review in Oh et al. (2015), the most common constraints (or, equivalently, controlled variables) in distributed formation control are of two types, viz. relative positions and distances which are physical variables that can often be readily sensed. However, though displacement- and distance-based methods are commonly used approaches in distributed formation control, in recent years, bearing-based formation control has also attracted research interest in part due to the advantage of using vision sensors without distance measurements (Trinh, Mukherjee et al., 2018; Trinh, Zhao et al., 2018; Tron, Thomas, Loianno, Daniilidis, & Kumar, 2016; Van Tran, Trinh, Zelazo, Mukherjee, & Ahn, 2018).

This paper especially focuses on distance-based formation control. Compared with displacement- and bearing-based formation control (Lee & Ahn, 2016; Nuno, Loria, Hernández, Maghenem, & Panteley, 2020; Trinh, Mukherjee, Zelazo & Ahn, 2018; Trinh, Zhao et al., 2018; Tron et al., 2016; Van Tran et al., 2018; Xiao, Wang, Chen, & Gao, 2009), distance-based formation control has the remarkable advantage that knowledge of a global (common) coordinate system (or, equivalently, agents' orientation information) is not required by each individual agent. This brings benefits for practical formation applications since agents do not require additional control laws or communication to share a common coordinate system. Consequently, this advantage has led to the development of distance-based formation control applications, such as formation shape control (Cortés, 2009; Krick, Broucke, & Francis, 2009; Sun, Mou, Anderson et al., 2016), formation flocking control (Deghat, Anderson, & Lin, 2015; Sun, Anderson et al., 2017; Sun, Mou, Deghat et al., 2016), formation maneuvering (Cai & de Queiroz, 2015; Mehdifar, Hashemzadeh, Baradarannia, & de Queiroz, 2018), etc.

However, even though such distance-based formation control has the aforementioned advantage, there are some critical and

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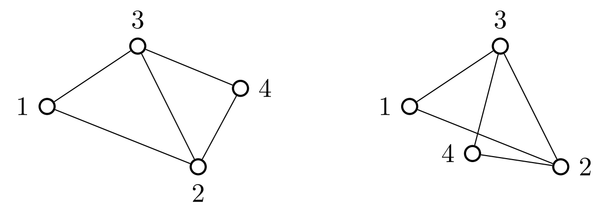
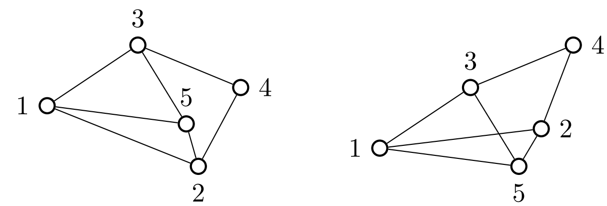
practical research issues when specifying a formation using distance constraints only. It has been observed in the literature that a formation specification, which will be formally defined later, composed of only the distance constraints may cause formation specification ambiguities, such as *flip and flex ambiguities* (Anderson, Yu, Fidan, & Hendrickx, 2008; De Queiroz, Cai, & Feemster, 2019) and *reflection ambiguity* (Anderson, Sun, Sugie, Azuma, & Sakurama, 2017; Kang, Park, & Ahn, 2016; Liu, Fernández-Kim, & de Queiroz, 2020). Flip and flex ambiguities mean that a formation specification is not adequate to specify a formation up to congruence (or equivalence under translation, rotation and reflection), i.e., the specifications are consistent with there being two (or more) formations which are not related by a common transformation of all agent positions by an element of the Euclidean group $E(2)$ (in two dimensions) or $E(3)$ (in three dimensions). In two dimensions, reflection ambiguities arise when there can be two formations which are not related by a common transformation of all agent positions by an element of $SE(2)$, though they may be related by a common transformation of all agent positions by an element of $E(2)$; this corresponds to there being two congruent transformations which differ by a reflection and not just a translation and a rotation. An equivalent statement holds when the ambient space is $\mathbb{R}^3$.

As examples, in Fig. 1(a), we can observe that there are non-congruent formation shapes consistent with the same distance constraints according to the position of agent 4, and a flip ambiguity exists. Fig. 1(b) shows a flex ambiguity, where agent 2 is flipped over the line described by agents 1 and 5, and, accordingly, the formation results in a different shape while all of the distance constraints are maintained. It is well known that the concept of global distance rigidity (Connelly, 2005) can be employed to remove the possibility of flip and flex ambiguities. However, the global rigidity concept still cannot guarantee a unique arrangement of agents under distance constraints due to the reflection ambiguity. For example, considering the two globally distance rigid formations consistent with the same distance constraints as shown in Fig. 1(c), the left formation includes the counterclockwise order of the agents to be 1, 2, 4, and 3 while its reflection has the order of 1, 3, 4, and 2; here, there is no flip/flex ambiguity but reflection ambiguity. This fact motivates us to focus on a research problem associated with characterizing a rigid formation and a specific arrangement of agents without such ambiguities.

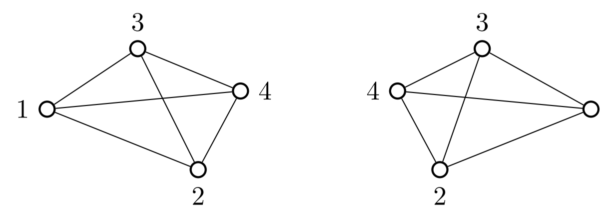
To carry the discussion further, at this point we make several definitions. An *n-agent formation* in 2-D space is a collection of n-points in $\mathbb{R}^2$. The *formation position* is the set of coordinates $p_i \in \mathbb{R}^2, i \in \{1, 2, \ldots, n\}$. A *formation shape*, for the purposes of this paper, is the equivalence class of all formations obtained by allowing a common transformation of each agent position by an element of $SE(2)$, i.e., the shape is the formation but without taking into account its position or orientation, and it remains constant if the formation itself translates or rotates. Thus, if $p = [p_1^\top, p_2^\top, \ldots, p_n^\top]^\top$ is the position of a formation, the associated formation shape is the equivalence class defined by the set of position vectors, i.e., $\{Lp_1, Lp_2, \ldots, Lp_n\}, \forall L \in SE(2)$. The formation shape does *not* remain constant if a reflection operation occurs (although such an operation would give rise to a congruent formation shape), but does remain constant if a rotation or translation occurs. A *formation specification* is a list of distance constraints involving agent pairs in the formation and, in this paper at least, signed areas involving triples of agents in the formation.[1] The inclusion of signed area constraints aims to eliminate reflection ambiguities as well as flip/flex ambiguities.



(a) Flip ambiguity



(b) Flex ambiguity



(c) Reflection ambiguity

Fig. 1. Example of flip, flex, and reflection ambiguities in 2-D, where the vertices and edges denote agents and distance constraints, respectively.

Note that a formation specification may or may not determine the formation shape uniquely. For example, different shapes consistent with the same formation specification might exist which are not congruent (see Figs. 1(a) and 1(b)), or are always congruent but differ by a reflection (see Fig. 1(c)). There may be a continuum of different shapes (such as when there are no area constraints, and the distance constraints are not sufficient to enforce distance rigidity in the usual sense), or there may be a finite number of different formation shapes; the latter case would arise if the distance constraints enforce only distance rigidity rather than global distance rigidity while one should impose additional distance or area constraints to eliminate reflection ambiguity. For convenience of analysis in the sequel, we will use the term *formation specification ambiguity* to denote the circumstance and say a formation specification is ambiguous where a formation specification does not determine a unique formation shape. If a formation specification does not allow any smooth motions of agents to deform a formation shape, then the n-agent formation is said to be *rigid*.[2]

It was observed in the literature (Anderson et al., 2017; Kang et al., 2016; Kwon, Sun, Anderson, & Ahn, 2020; Liu et al., 2020) that signed areas can contribute to the elimination of reflection ambiguity and indeed other formation specification ambiguities. For example, considering the formation in Fig. 1(a) again, two additional signed area constraints are imposed to determine a unique formation shape without formation specification ambiguities as shown in Fig. 2(a). In fact, signed area constraints can also dispense with redundant distance constraints while maintaining

[1] A signed area refers to an area with a positive or negative value according to two possible orientations in 2-D space.

[2] There are various rigidity theories to achieve rigid formations, depending on different types of constraints, in the literature (Buckley & Egerstedt, 2021; Cao, Li, & Xie, 2019; Chen, Cao, & Li, 2020; Hendrickson, 1992; Jing, Zhang, Lee, & Wang, 2019; Kwon & Ahn, 2020; Roth, 1981; Su, Hu, Li, & Chen, 2020; Zhao & Zelazo, 2016).

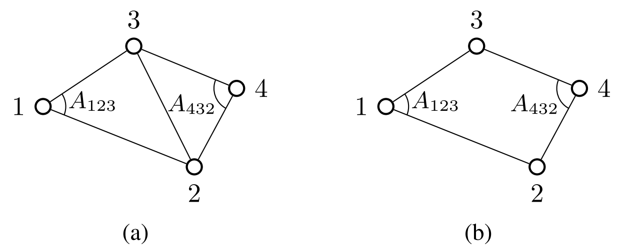


Fig. 2. Example of rigid formations with both distance- and signed area-constraints in 2-D, where the solid lines indicate distance constraints and the symbol A_{ijk} denotes a signed area constraint determined counterclockwise from $p_j - p_i$ to $p_k - p_i$.

the property of formation rigidity; for example, the formation in Fig. 2(b) is still rigid in spite of removing the distance constraint between agents 2 and 3 from the rigid formation in Fig. 2(a). To make use of this property of signed areas in formation rigidity, we consider the signed areas as additional constraints together with distance constraints in specifying rigid formations. The existing works (Anderson et al., 2017; Kang et al., 2016; Liu et al., 2020) have studied several special cases of formation specification control with signed constraints, including 3- and 4-agent formation systems (Anderson et al., 2017) and formation systems under specific directed sensing topologies (Kang et al., 2016; Liu et al., 2020). However, these works do not take into account formation rigidity. To the best of our knowledge, only our previous work (Kwon et al., 2020) deals with formation rigidity and formation specification control with distances and signed areas, which will be extended in this paper. The detail on the comparison with (Kwon et al., 2020) can be found in Remark 3.1.

The contributions of this paper are described as follows. First, we develop a rigidity theory with reference to the concept of hybrid rigidity introduced in Kwon et al. (2020), where the new rigidity theory studied in this paper is termed *sign rigidity theory*. The sign rigidity theory facilitates an examination of whether a multi-agent formation consistent with distance- and signed area-constraints is rigid or not. This theory includes three sub-concepts, i.e., *sign rigidity*, *global sign rigidity*, and *infinitesimal sign rigidity*. In particular, the concept of global sign rigidity has the property of eliminating all of the aforementioned formation specification ambiguities. To achieve formations with global sign rigidity, we introduce a *signed Henneberg construction*. This construction is a fundamental theoretical concept to grow formations while maintaining global sign rigidity, which is based on the *vertex addition* operation of the conventional Henneberg construction introduced in Eren, Anderson, Whiteley, Morse, and Belhumeur (2004) and Tay and Whiteley (1985). The relationship between the distance rigidity theory and sign rigidity theory is also established. Second, we apply the sign rigidity theory to formation specification control in 2-D space, where the control objective is to move a collection of agents to achieve the desired-distances and -signed areas given a formation specification. It is shown that if the signed Henneberg construction is employed to construct a target formation, then almost global stability is guaranteed for a proposed control law without any formation specification ambiguities. In particular, the advantage of distance-based formation control still remains in the proposed control law, that is, a global (common) coordinate system and coordinate frame orientation information of neighbor agents are not required.

The remaining parts of this paper are organized as follows. Section 2 introduces several notations and some background. In Section 3, the sign rigidity theory is developed. In Section 4, based on the sign rigidity theory, almost global stability of a formation control system with the proposed distributed control law is studied in 2-D space. Section 5 provides simulation results to support the main theories, and Section 6 finally concludes this paper.

2. Preliminary

This section briefly reviews some background on distance rigidity theory. First, several notations frequently used in this paper are as follows. We denote the Euclidean norm of a vector and the cardinality of a set by $\|\cdot\|$ and $|\cdot|$, respectively. The null space and the rank of a matrix are denoted as $\text{null}(\cdot)$ and $\text{rank}(\cdot)$, respectively. A symbol $(\cdot)^*$ denotes a desired value or vector with desired values for $(\cdot)$. We denote an undirected graph $\mathcal{G}$ by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ and $\mathcal{E}$ are the vertex set with $\mathcal{V} = \{1, 2, \ldots, n\}$ and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, respectively. An edge in $\mathcal{E}$ is represented by (i, j) for $i, j \in \mathcal{V}$, where it is assumed that $(i, j) = (j, i)$ for all $(i, j) \in \mathcal{E}$. The vertices in $\mathcal{V}$ are regarded as agents and the edge set $\mathcal{E}$ represents a formation specification consistent with a set of distance constraints. Then, a framework is defined as $(\mathcal{G}, p)$, where p denotes a realization of $\mathcal{G}$ given by $p = [p_1^\top, p_2^\top, \ldots, p_n^\top]^\top \in \mathbb{R}^{2n}$ for $p_i \in \mathbb{R}^2, i \in \mathcal{V}$.

2.1. Distance rigidity theory

The concept of distance rigidity is widely used to characterize rigid formations obeying prescribed distance constraints.

Definition 2.1 (*Distance Rigidity Asimow & Roth, 1978, 1979; Hendrickson, 1992; Roth, 1981*)**.** A framework $(\mathcal{G}, p)$ is distance rigid in $\mathbb{R}^2$ if there exists a neighborhood $\mathcal{U}_p \subseteq \mathbb{R}^{2n}$ of p such that each framework $(\mathcal{G}, q)$, $q \in \mathcal{U}_p$, equivalent[3] to $(\mathcal{G}, p)$ is congruent[4] to $(\mathcal{G}, p)$.

If any framework $(\mathcal{G}, q)$ equivalent to $(\mathcal{G}, p)$ is congruent to $(\mathcal{G}, p)$, then the framework $(\mathcal{G}, p)$ is said to be globally distance rigid in $\mathbb{R}^2$ (Connelly, 2005). Moreover, in the literature (Asimow & Roth, 1979; Hendrickson, 1992; Roth, 1981), the concept of infinitesimal distance rigidity distinguished from distance rigidity was introduced; the definition is omitted but we observe the following property.

Lemma 2.1. *(Asimow & Roth, 1979; Hendrickson, 1992; Roth, 1981) A framework $(\mathcal{G}, p)$ with $n \geq 2$ vertices in $\mathbb{R}^2$ is infinitesimally distance rigid if and only if $\text{rank}(R_d) = 2n - 3$, where R_d denotes the distance rigidity matrix defined as $R_d = \frac{\partial f_d}{\partial p}, f_d = \frac{1}{2}\left[\ldots, \|p_i - p_j\|^2, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{E}|}$ for $(i, j) \in \mathcal{E}$.*

The relationship between the distance rigidity and infinitesimal distance rigidity is established in Asimow and Roth (1979): one conclusion is that if a framework $(\mathcal{G}, p)$ is infinitesimally distance rigid in $\mathbb{R}^2$ then $(\mathcal{G}, p)$ is distance rigid in $\mathbb{R}^2$. We now introduce one more modification of the distance rigidity concept, which is introduced in Chen, Belabbas, and Başar (2017) as follows.

Definition 2.2 (*Strong Distance Rigidity*)**.** A distance rigid framework $(\mathcal{G}, p)$ is said to be strongly distance rigid if $p_j - p_i$ and $p_k - p_i$ for all $(i, j), (i, k) \in \mathcal{E}$ in 2-D space are linearly independent.

[3] Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are said to be equivalent if it holds that $\|p_j - p_i\| = \|q_j - q_i\|$ for all $(i, j) \in \mathcal{E}$.

[4] Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are said to be congruent if it holds that $\|p_j - p_i\| = \|q_j - q_i\|$ for all $i, j \in \mathcal{V}$.

3. Sign rigidity theory

This section develops a sign rigidity theory to achieve a rigid formation consistent with distance- and signed area-constraints. The signed areas can be considered as signed constraints with a real value and a sign as follows:

$$A_{ijk} = \frac{1}{2}\det\begin{bmatrix} p_j - p_i & p_k - p_i \end{bmatrix} = \frac{1}{2}\left\|p_j - p_i\right\|\left\|p_k - p_i\right\|\sin(\theta^i_{jk}),\ i,j,k \in \overline{\mathcal{V}}, \tag{1}$$

where $\theta^i_{jk} \in [0, 2\pi)$ denotes the signed angle measured counterclockwise from $p_j - p_i$ to $p_k - p_i$, and $\overline{\mathcal{V}}$ indicates a subset of $\mathcal{V}$ and contains those vertices which appear as a member of at least one vertex triple involving a signed area constraint. We then define a set $\mathcal{S}$ to denote a list of signed areas as follows:

$$\mathcal{S} = \{(i,j,k) \in \overline{\mathcal{V}}^3 | A_{ijk} \text{ for some } i,j,k \in \overline{\mathcal{V}}\}. \tag{2}$$

When adding the signed areas to a formation specification, a new framework is defined as a triplet of a graph $\mathcal{G}$, a set of signed areas $\mathcal{S}$, and a realization p, i.e., $(\mathcal{G}, \mathcal{S}, p)$. In this paper, this new framework is termed a *signed framework*.

Remark 3.1 (*Comparison with Kwon & Ahn, 2020; Kwon, Sun, Anderson, & Ahn, 2019; Kwon et al., 2020*)**.** In the publication (Kwon & Ahn, 2020), the generalized weak rigidity theory was introduced to specify rigid formations with distance- and subtended angle-constraints, where the subtended angles are denoted by cosine functions. The subtended angle constraint, however, is not suitable to handle formation specification ambiguities due to the fact that $\cos\theta^i_{jk} = \cos(2\pi - \theta^i_{jk})$, i.e., θ^i_{jk} cannot be distinguished from $2\pi - \theta^i_{jk}$. On the other hand, the signed area defined in (1) is able to distinguish θ^i_{jk} from $2\pi - \theta^i_{jk}$ by its sign, which contributes to the elimination of formation specification ambiguities.

In the works (Kwon et al., 2019, 2020), we employed normalized signed areas as signed constraints in a formation specification, and introduced the hybrid rigidity theory. On the other hand, in this paper, the normalized signed areas are replaced by signed areas without the normalization requirement. Strictly speaking, the definitions of signed constraints between this paper and the previous works are different. Thus, the sign rigidity theory studied in this paper should be distinguished from the hybrid rigidity theory. Moreover, this paper will explore a wide range of issues, ranging from formation rigidity to stability and equilibrium analysis of rigid formation systems, many of which were not studied in Kwon et al. (2019, 2020).

3.1. Sign rigidity

To introduce the concept of sign rigidity, we first need to define new notions in a similar way to the notions of equivalence and congruence in Definition 2.1.

Definition 3.1 (*Distance-Sign Equivalence*)**.** Two signed frameworks $(\mathcal{G}, \mathcal{S}, p)$ and $(\mathcal{G}, \mathcal{S}, q)$ with $\mathcal{E} \neq \emptyset$ and $\mathcal{S} \neq \emptyset$ are distance-sign equivalent if the following two conditions are satisfied:

- $\|p_j - p_i\| = \|q_j - q_i\|, \forall (i,j) \in \mathcal{E}$,
- $\left(A_{ijk}\right)_{\in(\mathcal{G},\mathcal{S},p)} = \left(A_{ijk}\right)_{\in(\mathcal{G},\mathcal{S},q)}, \forall (i,j,k) \in \mathcal{S}$,

where $(\cdot)_{\in(\mathcal{G},\mathcal{S},p)}$ and $(\cdot)_{\in(\mathcal{G},\mathcal{S},q)}$ denote the signed area terms belonging to $(\mathcal{G}, \mathcal{S}, p)$ and $(\mathcal{G}, \mathcal{S}, q)$, respectively.

Definition 3.2 (*Distance-Sign Congruence*)**.** Two signed frameworks $(\mathcal{G}, \mathcal{S}, p)$ and $(\mathcal{G}, \mathcal{S}, q)$ with $\mathcal{E} \neq \emptyset$ and $\mathcal{S} \neq \emptyset$ are distance-sign congruent if the following two conditions are satisfied:

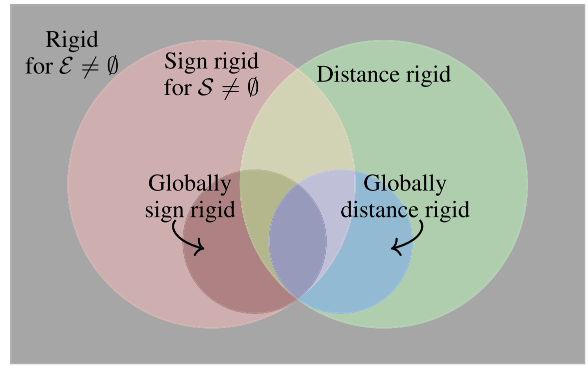


Fig. 3. Relation between (global) sign rigidity and (global) distance rigidity.

- $\|p_j - p_i\| = \|q_j - q_i\|, \forall i,j \in \mathcal{V}$,
- $\left(A_{ijk}\right)_{\in(\mathcal{G},\mathcal{S},p)} = \left(A_{ijk}\right)_{\in(\mathcal{G},\mathcal{S},q)}, \forall i,j,k \in \mathcal{V}$.

The concepts of sign rigidity and global sign rigidity are defined as follows.

Definition 3.3 (*Sign Rigidity*)**.** A signed framework $(\mathcal{G}, \mathcal{S}, p)$ is said to be sign rigid in $\mathbb{R}^2$ if there exists a neighborhood $\mathcal{B}_p \subseteq \mathbb{R}^{2n}$ of p such that each signed framework $(\mathcal{G}, \mathcal{S}, q)$ for $q \in \mathcal{B}_p$ which is distance-sign equivalent to $(\mathcal{G}, \mathcal{S}, p)$ is distance-sign congruent to $(\mathcal{G}, \mathcal{S}, p)$.

Definition 3.4 (*Global Sign Rigidity*)**.** A signed framework $(\mathcal{G}, \mathcal{S}, p)$ is said to be globally sign rigid in $\mathbb{R}^2$ if any signed framework $(\mathcal{G}, \mathcal{S}, q)$ which is distance-sign equivalent to $(\mathcal{G}, \mathcal{S}, p)$ is distance-sign congruent to $(\mathcal{G}, \mathcal{S}, p)$.

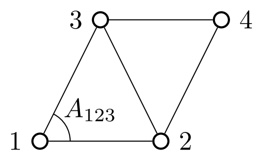
Based on the above definitions, we can establish a relationship between the (global) sign rigidity and (global) distance rigidity, which is shown in Fig. 3; some examples are also provided in Fig. 4. Note that, from Definition 3.4, a globally sign rigid framework has a unique formation shape[5] (up to a translation and a rotation of the entire formation) without any formation specification ambiguities.

We can observe that the signed areas can determine an arrangement of agents in formation characterization. For example, considering the framework in Fig. 4(a), under the given distance and area constraints, the position of agent 4 can be flipped over edge (2, 3) while the distances among the agents 2, 3 and 4 are maintained, whereas the agent 1 is fixed with reference to the agents 2 and 3 due to the signed area constraint A_{123}. Another example is that if the signed area A_{123} is excluded from the framework in Fig. 4(b), then the entire framework with only distances is still globally distance rigid but not globally sign rigid anymore, which implies that the whole framework can be allowed to be reflected. The entirely reflected framework changes the agents' ordering (clockwise or counterclockwise) for any triple even though the reflected framework is still globally distance rigid. In this regard, compared with the concept of (global) distance rigidity, the (global) sign rigidity concept can remove the possibility of reflection ambiguity as well as flip/flex ambiguities. A summary of the two rigidity theories and the associated ambiguity issues are shown in Table 1.

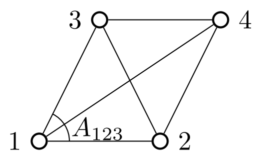
[5] Definition 3.4 does not allow a signed framework to involve a partial or overall reflection of the framework, which implies uniqueness of a signed framework being globally sign rigid (up to a translation and a rotation of the entire formation) without any formation specification ambiguities.

Table 1
The possible types of formation specification ambiguities and removable types among the possible ambiguities according to rigidity concepts. The removable ambiguity issues indicate those issues that each rigidity concept can eliminate with appropriately chosen constraints.

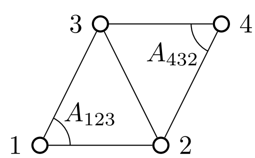
Rigidity	Possible ambiguity issues	Removable ambiguity issues
Distance rigidity	Flip/flex ambiguities & reflection ambiguity	Flip/flex ambiguities
Global distance rigidity	Reflection ambiguity	N/A
Sign rigidity	Flip/flex ambiguities & reflection ambiguity	Flip/flex ambiguities & reflection ambiguity
Global sign rigidity	N/A	N/A



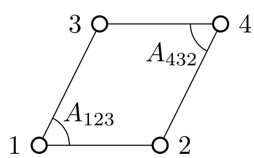
(a) Sign and distance rigid, but not globally sign and distance rigid.



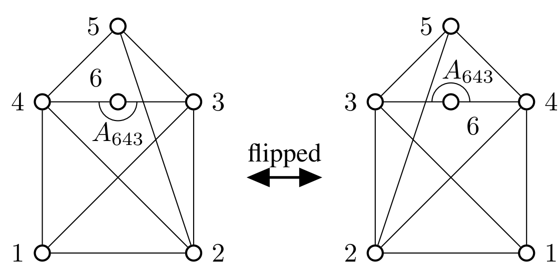
(b) Globally sign and distance rigid.



(c) Distance rigid and globally sign rigid, but not globally distance rigid.



(d) Sign rigid, but not distance rigid. If $\theta_{23}^1 = \theta_{32}^4 = \frac{\pi}{2}$ (or, equivalently, $\sin(\theta_{23}^1) = \sin(\theta_{32}^4) = 1$), then this framework is globally sign rigid.



(e) Sign rigid and globally distance rigid, but not globally sign rigid, where θ_{43}^6 is set as π.

Fig. 4. Formation examples to show the relation in Fig. 3, where the solid lines denote distance constraints.

3.2. Infinitesimal sign rigidity

In this subsection, we introduce the concept of infinitesimal sign rigidity, which can be viewed as an extended version of infinitesimal distance rigidity. In comparison with the sign rigidity concept, the remarkable property of infinitesimal sign rigidity is that we can examine whether or not a signed framework is rigid in an algebraic manner.

We first define several functions and notations. For a signed framework $(\mathcal{G}, \mathcal{S}, p)$, the *signed rigidity function* $F_d^s : \mathbb{R}^{2n} \to \mathbb{R}^{(|\mathcal{E}|+|\mathcal{S}|)}$ is defined as follows:

$$F_d^s(p) = \begin{bmatrix}\mathbf{D}^\top & \mathbf{S}^\top\end{bmatrix}^\top, \tag{3}$$

where $\mathbf{D} = \frac{1}{2}\left[\ldots, \|p_j - p_i\|^2, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{E}|}$ for $(i,j) \in \mathcal{E}$ and $\mathbf{S} = \left[\ldots, A_{ijk}, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{S}|}$ for $(i,j,k) \in \mathcal{S}$. Assuming smooth motions of the framework $(\mathcal{G}, \mathcal{S}, p)$ while maintaining all constraints, we have the following time derivative of (3):

$$\dot{F}_d^s = R_d^s \dot{p} = 0, \tag{4}$$

where R_d^s denotes the *signed rigidity matrix* given by

$$R_d^s(p) = \frac{\partial F_d^s(p)}{\partial p} = \begin{bmatrix}\frac{\partial \mathbf{D}}{\partial p} \\ \frac{\partial \mathbf{S}}{\partial p}\end{bmatrix} = \begin{bmatrix}R_d \\ R_s\end{bmatrix} \in \mathbb{R}^{(|\mathcal{E}|+|\mathcal{S}|)\times 2n}, \tag{5}$$

and $\dot{p}$ is called infinitesimal motions of $(\mathcal{G}, \mathcal{S}, p)$ that preserve the constraints in the rigidity function (3). With reference to Sun, Park et al. (2017, Lemma 1), a basis for rigid transformations is denoted by

$$L_p = \{\mathbb{1}_n \otimes I_2, (I_n \otimes J)p\}, \tag{6}$$

where the symbol $\mathbb{1}_n$ denotes an all-ones vector, $\mathbb{1}_n = [1, \ldots, 1]^\top \in \mathbb{R}^n$, and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$. The signed rigidity matrix has the following property.

Lemma 3.1. *It holds that* $\operatorname{span}\{L_p\} \subseteq \operatorname{null}(R_d^s(p))$ *for a signed framework* $(\mathcal{G}, \mathcal{S}, p)$ *in* $\mathbb{R}^2$*, which implies that* $\operatorname{rank}\left(R_d^s(p)\right) \le 2n-3$.

The proof of this lemma can be completed in the same way as our previous work (Kwon et al., 2019). We are now ready to define and explore the concept of infinitesimal sign rigidity. The definition of infinitesimal sign rigidity is as follows.

Definition 3.5 (*Infinitesimal Sign Rigidity*)**.** A signed framework $(\mathcal{G}, \mathcal{S}, p)$ is said to be infinitesimally sign rigid if all infinitesimal motions of $(\mathcal{G}, \mathcal{S}, p)$ correspond to only a translation and a rotation of the entire framework.

The following result shows the necessary and sufficient condition for a signed framework to be infinitesimally sign rigid.

Theorem 3.1. *A signed n-agent framework* $(\mathcal{G}, \mathcal{S}, p)$ *is infinitesimally sign rigid in* $\mathbb{R}^2$ *if and only if* $\operatorname{rank}\left(R_d^s(p)\right) = 2n-3$.

Proof. It is observed that $(\mathbb{1}_n \otimes I_2)$ and $(I_n \otimes J)p$ respectively correspond to a translation and a rotation of the entire formation. Moreover, it follows from Lemma 3.1 that $\operatorname{span}\{\mathbb{1}_n \otimes I_2, (I_n \otimes J)p\} \subseteq \operatorname{null}(R_d^s(p))$. The condition of $\operatorname{rank}\left(R_d^s(p)\right) = 2n-3$ indicates $\operatorname{span}\{\mathbb{1}_n \otimes I_2, (I_n \otimes J)p\} = \operatorname{null}(R_d^s(p))$ with a dimension of three, which implies that all infinitesimal motions of $(\mathcal{G}, \mathcal{S}, p)$ correspond to only a translation and a rotation of the entire framework. Therefore, this proof directly follows from Definition 3.5. ■

In the following, we establish a relationship between the concepts of strong distance rigidity, infinitesimal distance rigidity, and infinitesimal sign rigidity, which will be essentially used to analyze almost global stability in the next section. The following proposition shows a relationship between strong distance rigidity and infinitesimal distance rigidity.

Proposition 3.1 (*Chen et al., 2017, Proposition 1*)**.** *If a framework* $(\mathcal{G}, p)$ *is strongly distance rigid in* $\mathbb{R}^2$*, then* $(\mathcal{G}, p)$ *is infinitesimally distance rigid in* $\mathbb{R}^2$.

We then have the following corollary.

Corollary 3.1. *If a framework $(\mathcal{G}, p)$ is strongly distance rigid in $\mathbb{R}^2$, then the signed framework $(\mathcal{G}, \mathcal{S}, p)$ including the same distance constraints as $(\mathcal{G}, p)$ is infinitesimally sign rigid in $\mathbb{R}^2$.*

Proof. It follows from Proposition 3.1 that there exists a nonzero $(2n-3)\times(2n-3)$ minor of R_d, which implies from the definition $R_d^s(p) = \begin{bmatrix} R_d \\ R_s \end{bmatrix}$ that there exists a nonzero $(2n-3)\times(2n-3)$ minor of R_d^s. Then, this proof directly follows from Theorem 3.1. ■

In addition, from Proposition 3.1 and Corollary 3.1 (including the proofs), we can observe that if a framework $(\mathcal{G}, p)$ is infinitesimally distance rigid in $\mathbb{R}^2$, then the signed framework $(\mathcal{G}, \mathcal{S}, p)$ is infinitesimally sign rigid.

3.3. Signed Henneberg construction

As studied in the previous subsections, although additional area constraints can render a signed framework to be sign rigid or infinitesimally sign rigid, there may occur formation specification ambiguities. Moreover, as shown in the definition of signed area in (1), the signed areas can cause a new ambiguity issue; for example, considering the sign rigid signed framework in Fig. 4(d) with $\|p_2 - p_1\| = \|p_3 - p_1\| = \|p_3 - p_2\| = \|p_4 - p_2\| = 1$ and $A_{123} = A_{432} = \sqrt{3}/4$, the values of subtended angles θ_{23}^1 and θ_{32}^4 can be either $2\pi/3$ or $\pi/3$, which leads to two different formation shapes and is similar to the sine ambiguity issue introduced in Kwon et al. (2019, 2020). The sine ambiguity arises when signed area constraints involving sine functions are not uniquely determined due to the fact that $\sin(\alpha) = \sin(\pi - \alpha)$ for $\alpha \in [0, \pi]$. Hence, to eliminate the possibility of such ambiguity issues, we need to specify globally sign rigid frameworks with appropriately chosen constraints. In the following, we introduce a method to characterize a globally sign rigid framework without any formation specification ambiguities.

We consider a technique that combines the *vertex addition* of the conventional Henneberg construction introduced in Eren et al. (2004) and Tay and Whiteley (1985) with the signed area constraints. The technique leads to a new construction termed *signed Henneberg construction.*

Remark 3.2. The signed Henneberg construction employs some identical steps and some different steps as compared with the conventional Henneberg construction. The conventional Henneberg construction is a well-known approach to grow minimally distance rigid formations (Anderson et al., 2008; Eren et al., 2004). On the other hand, the signed Henneberg construction is employed to grow globally sign rigid formations.

The operation of the signed Henneberg construction is as follows. It is assumed at first that an initial signed framework satisfies $|\mathcal{V}| = 3$, $|\mathcal{E}| = 3$ and $|\mathcal{S}| = 1$. For a given globally sign rigid framework $(\mathcal{G}, \mathcal{S}, p)$, an agent ν is added to $(\mathcal{G}, \mathcal{S}, p)$ in order that the combined framework $(\bar{\mathcal{G}}, \bar{\mathcal{S}}, \bar{p})$ is composed of triangular frameworks with additional 2 distance- and 1 signed area-constraints such that $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, $\bar{\mathcal{V}} = \mathcal{V} \cup \{\nu\}$, $\bar{\mathcal{E}} = \mathcal{E} \cup \{(i,\nu), (j,\nu)\}$, $\bar{\mathcal{S}} = \mathcal{S} \cup \{(\bar{i}, \bar{j}, \bar{k})\}$, and $\bar{p} = [p^\top, p_\nu^\top]^\top \in \mathbb{R}^{2(n+1)}$ for $i, j \in \mathcal{V}$, $\bar{i}, \bar{j}, \bar{k} \in \{i, j, \nu\}$ and $\bar{i} \neq \bar{j} \neq \bar{k}$; an example of signed Henneberg construction is described in Fig. 5.[6]

[6] Note that it is possible to dispense with the magnitude value defining the signed area of a particular extension and just use its sign in the signed Henneberg construction. There will be an operation using signs only that leads to a unique extension similarly to the signed Henneberg construction. However, it is impossible to retain the real value and sign, and dispense with redundant distance constraints, as in general this will introduce an ambiguity. If in the second formation in Fig. 5, the distance constraint $(3, 4)$ was not given, knowledge of the area would determine the sign of the angle θ_{32}^4, but the magnitude is undetermined up to a binary ambiguity, and accordingly the distance constraint $(3, 4)$ is similarly subject to an ambiguity.

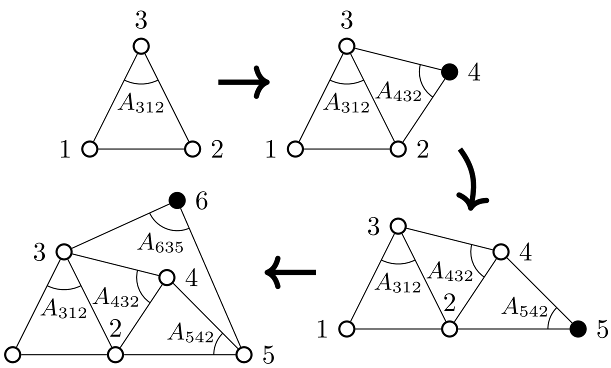


Fig. 5. Example of signed Henneberg construction, where the solid lines indicate the distance constraints. The added agent in each step is denoted by a black circle.

We can merge the signed Henneberg construction with the concept of infinitesimal sign rigidity to achieve a globally sign rigid framework, which is shown in the following proposition.

Proposition 3.2. *A signed framework $(\mathcal{G}, \mathcal{S}, p)$ is globally sign rigid if $(\mathcal{G}, \mathcal{S}, p)$ is infinitesimally sign rigid and has the signed Henneberg construction.*

Proof. It follows from Theorem 3.1 that if $(\mathcal{G}, \mathcal{S}, p)$ is infinitesimally sign rigid then $\operatorname{rank}\left(R_d^s(p)\right) = 2n - 3$. Thus, $R_d^s(p)$ is of maximum row rank due to the fact of Lemma 3.1, which implies similarly to Asimow and Roth (1978, Proposition 2) that there exists a neighborhood $\mathcal{U}_p$ such that $F_d^{s-1}(F_d^s(p)) \cap \mathcal{U}_p$ is a 3-dimensional smooth manifold, i.e., a set of 2-D rigid transformations. According to Definition 3.3, we conclude that $(\mathcal{G}, \mathcal{S}, p)$ is sign rigid.

Further note that a formation shape and an arrangement of agents with the signed Henneberg construction are uniquely determined since each agent in each sequence is uniquely positioned by 2 distance constraints and 1 area constraint as shown in Fig. 5. Hence, it follows from Definition 3.4 that $(\mathcal{G}, \mathcal{S}, p)$ is globally sign rigid. ■

The other operation of the conventional Henneberg construction, i.e., the *edge splitting* operation, also can be a candidate to be an operation of signed Henneberg construction, which can lead to the same result as Proposition 3.2; this work is omitted in this paper due to the space limitation.

4. Formation specification control based on the sign rigidity theory in 2-D space

This section applies the sign rigidity theory to formation specification control in 2-D space, where the control objective is to achieve desired distances and signed areas for a formation specification. In particular, it is assumed in this section that all frameworks are characterized via the signed Henneberg construction; thus a globally sign rigid framework is achieved when a signed framework is infinitesimally sign rigid as studied in Section 3.3. This section splits the formation specification control into two parts according to different agent dynamics: the first one is the case where each agent is governed by the single-integrator model; the other one is that each agent is assumed to be the unicycle model.

4.1. Under a single integrator-based system

In this subsection, a control system is introduced with an assumption that each agent is a single integrator. First, let us define several notations in the main control system law. The vectors composed of distance constraints and their desired values are defined as follows:

$$f_d^s(p) = \frac{1}{2}\left[\ldots, \|p_j - p_i\|^2, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{E}|}, \tag{7}$$

$$f_d^{s*} = \frac{1}{2}\left[\ldots, \|p_j - p_i\|^{*2}, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{E}|} \tag{8}$$

for $(i, j) \in \mathcal{E}$. Similarly, the vectors constituting signed areas and their desired values are denoted by

$$f_a^s(p) = K\left[\ldots, A_{ijk}, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{S}|}, \tag{9}$$

$$f_a^{s*} = K\left[\ldots, A_{ijk}^*, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{S}|} \tag{10}$$

for $(i, j, k) \in \mathcal{S}$, where K denotes a positive constant. We then define the formation specification error as

$$e(p) = \left[f_d^{s\top}(p) \quad f_a^{s\top}(p)\right]^\top - \left[f_d^{s*\top} \quad f_a^{s*\top}\right]^\top. \tag{11}$$

The main control system is derived based on the gradient flow law (Sakurama, Azuma, & Sugie, 2015) as follows:

$$\dot{p} = u = -\nabla\phi = -\overline{R}_d^{s\top}(p)e(p), \tag{12}$$

where $\phi = \frac{1}{2}e^\top(p)e(p)$ and $\overline{R}_d^s = \left[R_d^\top \quad KR_s^\top\right]^\top$. Note that the coefficient K does not have an effect on the rank property of rigidity matrix; that is, the matrix $\overline{R}_d^s$ has the same rank as the signed rigidity matrix R_d^s. Based on the structure of control law (12), a sensing topology can be defined for the law (12) to be distributed as follows.

Definition 4.1 (*Sensing Topology*)**.** The sensing topology for the control system (12) follows the undirected graph $\mathcal{G}^m = (\mathcal{V}, \mathcal{E}^m)$, where $\mathcal{E}^m = \{(i,j), (i,k), (j,k) | (i,j) \in \mathcal{E} \vee (i,j,k) \in \mathcal{S}\}$. The sensing directions are defined as bidirectional for $(i,j) \in \mathcal{E}^m$, and inter-agent relative positions are measured.

Considering the example (B.3) in Appendix B, we can check that only inter-agent relative positions are involved in control for each agent. The control law (12) follows the sensing topology defined in Definition 4.1, which enables distributed implementation of the control law (12), and does not require a global (common) coordinate system and coordinate frame orientation information of neighbor agents. In this sense, a target formation for the control law (12) always has trivial motions, such as a translation and a rotation of the entire formation. To handle the trivial motions, a modified control law or additional control laws may be required.

We remark that although a formation generated by the signed Henneberg sequences has a unique formation shape and arrangement of agents, it does not mean that the control system (12) has a unique equilibrium. In what follows, we will analyze a set of stable equilibrium points. First, we prove in the following theorem that if a target formation is infinitesimally sign rigid, then there exists a stable equilibrium point of the system (12).

Proposition 4.1. *Let ψ denote a set of realizations p associated with infinitesimally sign rigid frameworks that are distance-sign equivalent to a target signed framework. Then, under the control system (12), there exists a neighborhood $\mathcal{B}_{p^*}$ of p^* for any $p^* \in \psi$ such that an initial point $p(0) \in \mathcal{B}_{p^*}$ converges to a fixed point $p^\dagger \in \psi$ exponentially fast.*

The proof of Proposition 4.1 can be achieved by the center manifold theory; we refer the readers to Cao et al. (2019), Jing et al. (2019) and Kwon et al. (2019). As a matter of fact, it has been observed that the coefficient K in the system (12) has an effect on the existence of undesired but stable equilibrium points of the system (12). Based on the works (Anderson et al., 2017; Sugie, Tong, Anderson, & Sun, 2020), one can expect that if no weighting coefficient or a very small coefficient K is given to the signed areas, in comparison to terms reflecting distance errors, there will be a stable equilibrium point with incorrect orientation. For example, in Fig. 6, we can check different convergence properties of formations according to the value of K even though the formations have the same initial conditions and target formation to which they converge, i.e., the same initial positions, target distances, and target signed areas. As shown in Fig. 6, the initial agents in Fig. 6(a) converge to an undesired framework/equilibrium with a small K while the initial agents in Fig. 6(b) converge to the desired framework/equilibrium with a sufficiently large K. In Anderson et al. (2017), the discussion on K is provided for the specific case of 3- and 4-agent triangulated frameworks in $\mathbb{R}^2$. However, the formation control system introduced in this paper deals with a larger number of agents than that of Anderson et al. (2017); thus the result and technique in Anderson et al. (2017) cannot be directly employed in our work. We instead make use of the sign rigidity theory and the concept of strong rigidity to analyze the effect of K.

Proposition 4.1 establishes that if a target signed framework is infinitesimally sign rigid then there exists a stable equilibrium point of the system (12). Based on this fact, we next explore whether or not all stable equilibrium points of the system (12) satisfy the same distances and signed areas as the target signed framework for a sufficiently large K: this will in fact be shown in Theorem 4.1. To prove this, we need to state several useful facts. The following proposition shows a relation between the stability and strong distance rigidity.

Proposition 4.2. *Consider a signed framework $(\mathcal{G}, \mathcal{S}, p)$ established via a signed Henneberg construction. If $p = p^*$ is a stable equilibrium of the system (12) with sufficiently large K, then $(\mathcal{G}, p^*)$ is strongly distance rigid.*

Proof. See Appendix A.

What we have studied so far in this paper leads to the following result.

Corollary 4.1. *Under the same hypotheses as Proposition 4.2, the signed framework $(\mathcal{G}, \mathcal{S}, p^*)$ is infinitesimally sign rigid.*

Proof. It follows directly from Proposition 4.2 that $(\mathcal{G}, p^*)$ is strongly distance rigid if $p = p^*$ is stable for sufficiently large K. Moreover, we have that $(\mathcal{G}, \mathcal{S}, p^*)$ is infinitesimally sign rigid if $(\mathcal{G}, p^*)$ is strongly distance rigid from Corollary 3.1. ■

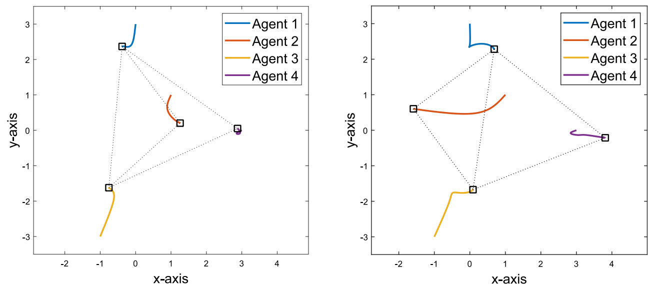
The following lemma shows that if K is sufficiently large then there are no undesired stable equilibrium points with incorrect signs of signed areas.

Lemma 4.1. *For sufficiently large K, stable equilibrium points of the system (12) occur only when the signs of signed areas in a signed framework $(\mathcal{G}, \mathcal{S}, p)$ are correct, i.e., the signs of signed areas are the same as the signs of desired signed areas.*

Proof. See Appendix C.

With the results of Corollary 4.1 and Lemma 4.1, we can finally conclude that if K is sufficiently large, then there are no undesired stable equilibrium points and further almost global stabilization is achieved; this is shown in the following theorem.

Theorem 4.1 (*Almost Global Convergence*)**.** *Suppose that a target signed framework is infinitesimally sign rigid and established via a signed Henneberg construction. Under the control system (12) for almost all initial conditions with sufficiently large K, all agents converge to a signed framework which is distance-sign congruent to the target signed framework.*



(a) $K = 0.3$ (b) $K = 2$

Fig. 6. Simulations on trajectories of 4 agents with different values of K under the controller (12) in $\mathbb{R}^2$, where the symbol □ denotes the final position for each agent, and the desired constraints are chosen as $\|p_2 - p_1\|^{*2} = \|p_3 - p_2\|^{*2} = 8$, $\|p_3 - p_1\|^{*2} = \|p_4 - p_1\|^{*2} = \|p_4 - p_3\|^{*2} = 16$, $A^*_{231} = 4\sin(\frac{\pi}{2})$ and $A^*_{413} = 8\sin(\frac{\pi}{3})$.

Proof. First, we can conclude from Proposition 4.2 and Corollary 4.1 that all frameworks at stable equilibrium points of the system (13) are strongly distance rigid and infinitesimally sign rigid for sufficiently large K. From Proposition 4.1, we also have the fact that if a target signed framework is infinitesimally sign rigid then there exists a stable equilibrium point of the system (12). In the following, we show that all signed frameworks at stable equilibrium points satisfy the same distances and signed areas as the target signed framework for sufficiently large K.

Consider the system (12) rewritten with respect to the definition $\overline{R}^s_d = \begin{bmatrix} R_d^\top & KR_s^\top \end{bmatrix}^\top$ as follows:

$$\dot{p} = -\overline{R}_d^{s\top}(p)e(p) = -R_d^\top(p)e_d(p) - KR_s^\top(p)e_a(p),$$
$$= -R_d^\top(p)e_d(p) - K^2R_s^\top(p)\bar{e}_a(p), \tag{13}$$

where $e_d(p) = f_d^s(p) - f_d^{s*}$, $e_a(p) = f_a^s(p) - f_a^{s*}$ and $\bar{e}_a(p) = \left[\ldots, A_{ijk} - A^*_{ijk}, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{S}|}$ for $(i,j,k) \in \mathcal{S}$. It is obvious that, at a stable equilibrium point p^*, the two terms $R_d^\top(p)e_d(p)$ and $R_s^\top(p)\bar{e}_a(p)$ in (13) are bounded. Then, we can observe that there exists a sufficiently large K such that $(13)|_{p=p^*} = -R_d^\top(p^*)e_d(p^*) - K^2R_s^\top(p^*)\bar{e}_a(p^*) \neq 0$ if $R_s^\top(p^*)\bar{e}_a(p^*) \neq 0$; however, this is a contradiction with $(13)|_{p=p^*} = 0$. Thus, to satisfy $(13)|_{p=p^*} = 0$ for sufficiently large K, it must hold that

$$R_d^\top(p^*)e_d(p^*) = R_s^\top(p^*)\bar{e}_a(p^*) = 0. \tag{14}$$

Moreover, since $(\mathcal{G}, p^*)$ is strongly distance rigid, we have that $(\mathcal{G}, p^*)$ is infinitesimally distance rigid from Proposition 3.1, i.e., $\text{rank}(R_d(p^*)) = 2n - 3$. Thus, the equality $R_d^\top(p^*)e_d(p^*) = 0$ in (14) directly leads to $e_d(p^*) = 0$ based on the fact that the target framework has the signed Henneberg construction with $|\mathcal{E}| = 2n - 3$ and $R_d(p^*)$ is of full row rank. Furthermore, it follows from Lemma 4.1 that, for sufficiently large K, a signed framework has the correct signs of the signed areas at p^*, which means that, since $e_d(p^*) = 0$, all signed areas are also satisfied, i.e., $e_a(p^*) = 0$. Thus, we conclude that a signed framework at a stable equilibrium point satisfies all desired distances and signed areas for sufficiently large K.

Let us next consider the Lyapunov function $V = \frac{1}{2}e^\top e$, and its time derivative along the trajectories of $\dot{e} = -\overline{R}^s_d\overline{R}_d^{s\top}e$:

$$\dot{V} = e^\top\dot{e} = -e^\top\overline{R}^s_d\overline{R}_d^{s\top}e = -\left\|\overline{R}_d^{s\top}e\right\|^2 \leq 0. \tag{15}$$

As we can see from (15), it holds that $\dot{V} = 0$ if and only if p belongs to the set of equilibrium points of the control system (12), which implies that all agents globally asymptotically converge to the set of equilibrium points. It has been proved in this argument that all stable formations are those formations constituting desired-distances and -signed areas for sufficiently large K. As a result, under the system (12) with sufficiently large K, if $p(0)$ does not lie on a stable manifold of a saddle point and the set of equilibria then $p(0)$ converges to a stable, strongly distance rigid, and infinitesimally sign rigid framework which is distance-sign congruent to the target signed framework. ■

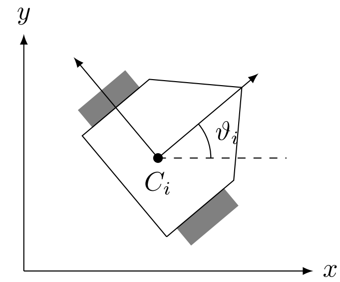


Fig. 7. Schematic of the unicycle model, where the symbol C_i denotes the center of mass for agent $i \in \mathcal{V}$.

It is challenging to analyze all equilibrium points of the formation specification control system (12) since there are multiple equilibrium points other than the desired equilibrium points. This is why we proposed the almost global attractiveness of the target signed frameworks with a sufficiently large K.

4.2. Under a unicycle model-based system

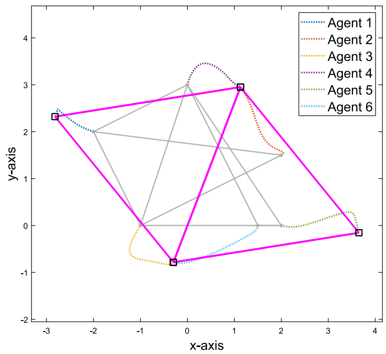
In this subsection, it is shown that the single integrator-based system can be extended to a system for nonholonomic models with reference to the work in Zhao, Dimarogonas, Sun, and Bauso (2017). In general, the control for unicycle models involves either position-only control or position and orientation control. This paper considers a position-only control law for a unicycle model-based system. That is, a formation shape is defined via position coordinates so that a target formation shape is realized when the position coordinates for all unicycle agents satisfy the formation specifications, while unicycle's heading angle variables are irrelevant for the target formation shape. The unicycle model for each agent is depicted in Fig. 7, and the dynamics is given by

$$\dot{x}_i = v_i\cos(\vartheta_i),$$
$$\dot{y}_i = v_i\sin(\vartheta_i),$$
$$\dot{\vartheta}_i = w_i, \tag{16}$$

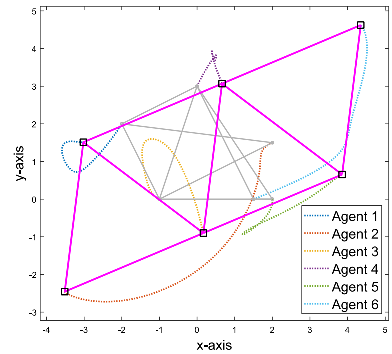
where, for agent $i \in \mathcal{V}$, x_i and y_i denote the position such that $p_i = [x_i, y_i]^\top$, and ϑ_i, v_i and w_i are the heading angle, linear velocity and angular velocity, respectively. Then, the control law is given as follows:

$$v_i = [\cos(\vartheta_i), \sin(\vartheta_i)]u_i,$$
$$w_i = [-\sin(\vartheta_i), \cos(\vartheta_i)]u_i, \tag{17}$$

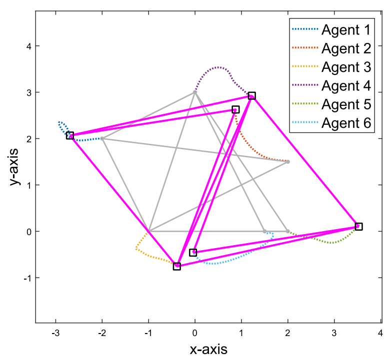
where u_i is the control input in the control law (12) for agent $i \in \mathcal{V}$. Just as the control law (12) is distributed, the system (17) is also distributed. Note that although the heading angle ϑ_i is an expression based on a global reference frame, it can be determined without global information in the control law (17). That is, an initial heading angle for a unicycle agent can be given according to its local reference frame. This is because there is no desired heading angle and the heading angle variables do not have an effect on convergence for the introduced control law (17), which will be shown in the proof of Theorem 4.2. It is remarkable that the control laws (12) and (17) can be modified to guarantee collision avoidance between agents. The expression of ϕ in (12) and (17) is invariant to a translation and a rotation of the entire formation, and the variants on ϕ can lead to adjustments to avoid excessive closeness of agents. To guarantee collision



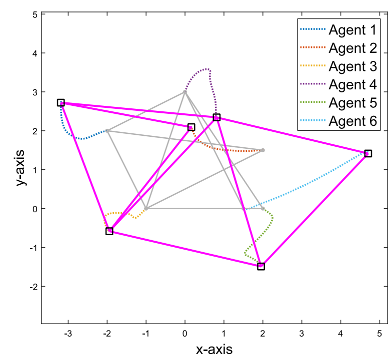
(a) Trajectories of agents under the control law with only distance constraints introduced in [11,40], where $\|p_j - p_i\|^* = 4$ for $(i, j) \in \mathcal{E}$ and $K = 0$.



(b) Trajectories of agents under the proposed controller (12) with $\|p_j - p_i\|^* = 4$ for $(i, j) \in \mathcal{E}$, $A^*_{ijk} = 4\sqrt{3}$ for $(i, j, k) \in \mathcal{S}$ and $K = 2$.



(c) Trajectories of agents under the proposed controller (12) with $\|p_j - p_i\|^* = 4$ for $(i, j) \in \mathcal{E}$, $A^*_{ijk} = 4\sqrt{3}$ for $(i, j, k) \in \mathcal{S}$ and $K = 0.4$.



(d) Trajectories of agents under the proposed controller (12) with $\|p_j - p_i\|^* = 4$ for $(i, j) \in \mathcal{E}$, $A^*_{ijk} = 4\sqrt{3}$ for $(i, j, k) \in \mathcal{S}$ and $K = 0.6$.

Fig. 8. Simulations on convergence to specific formation shapes under different control laws with the same initial positions in $\mathbb{R}^2$, where Figs. 8(b)–8(d) are under the same control law with the same desired constraints except for different values of K. The dashed line and the symbol $\square$ denote the trajectory and the final position for each agent, respectively, and the formations are characterized in accordance with $\mathcal{E} = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4), (3, 5), (4, 5), (4, 6), (5, 6)\}$ and $\mathcal{S} = \{(1, 2, 3), (3, 4, 1), (4, 3, 5), (5, 6, 4)\}$.

avoidance between agents, we can modify the function ϕ with an additional term for collision avoidance as studied in Dimarogonas and Johansson (2010), Dimarogonas, Loizou, Kyriakopoulos, and Zavlanos (2006).

We now prove that all unicycle agents converge to a signed framework in the same attraction region as the system (12) as follows.

Theorem 4.2. *Suppose that a target signed framework is infinitesimally sign rigid and established via a signed Henneberg construction. Then, under the control law (17) for almost all initial conditions with sufficiently large K, all agents converge to a signed framework which is distance-sign congruent to the target signed framework.*

Proof. Substituting the control law (17) into the dynamics (16) yields the following system:

$$\dot{p} = MM^\top u, \tag{18}$$

$$\dot{h} = M^\perp (M^\perp)^\top u, \tag{19}$$

where $M = \mathrm{diag}(h_1, \ldots, h_n) \in \mathbb{R}^{2n\times n}$, $M^\perp = \mathrm{diag}(h_1^\perp, \ldots, h_n^\perp) \in \mathbb{R}^{2n\times n}$, $h = [h_i^\top, \ldots, h_n^\top]^\top \in \mathbb{R}^{2n}$, and $h_i = [\cos(\vartheta_i), \sin(\vartheta_i)]^\top$ and $h_i^\perp = [-\sin(\vartheta_i), \cos(\vartheta_i)]^\top$. Then, consider the Lyapunov function $V = \frac{1}{2} e^\top e$ and its time derivative as follows:

$$\begin{aligned}\dot{V} = e^\top \dot{e} = e^\top \overline{R}^s_d \dot{p} &= -e^\top \overline{R}^s_d MM^\top \overline{R}^{s\top}_d e \\ &= -\left\| M^\top \overline{R}^{s\top}_d e \right\|^2 \\ &\le 0. \end{aligned} \tag{20}$$

To satisfy $\dot{V} = 0$, we need either (i) the case where p belongs to the set of equilibrium points of the control system (12), i.e., $u_i = 0$ for $\forall i \in \mathcal{V}$ or (ii) the case where $h_i \perp u_i$ for $u_i \neq 0$, $i \in \mathcal{V}$. It is clear that the second case cannot occur for $u_i \neq 0$, $i \in \mathcal{V}$. Therefore, in the same way as Theorem 4.1, we can conclude that all agents under the law (17) converge to a signed framework in the same attraction region as the system (12). ■

Similarly to the work (Zhao et al., 2017), the initial heading information for each agent is irrelevant to convergence and the final heading information is not determined by the control (17).

5. Simulation results

In this section, we provide several simulation results to validate the statements on the control systems (12) and (17). In the simulations, each agent only needs relative position measurements of its neighbors in its local coordinate system for distributed specification control. Under the unicycle model-based system, each agent additionally requires heading angle information in its local coordinate system.

5.1. Under the single integrator-based system

This subsection provides simulation results based on the single integrator-based system (12). A target formation in the simulations is chosen to be infinitesimally distance rigid or infinitesimally sign rigid in $\mathbb{R}^2$. Fig. 8 shows four different convergence

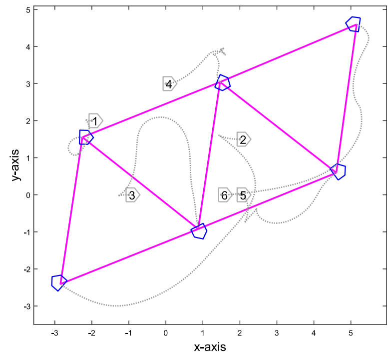


Fig. 9. Trajectories of unicycle agents under the controller (17), where all initial and desired conditions are the same as those in Fig. 8(b), and the dashed line denotes the trajectory of each unicycle agent.

outcomes under the conventional control law (Chen et al., 2017; Sun, Mou, Anderson & Cao, 2016) and the control law (12). In Fig. 8(b), it is shown that, under the control law (12) with the formation constraints defined using the signed Henneberg construction, the initial agents converge to a signed framework which is distance-sign congruent to the target signed framework without any formation specification ambiguities as studied in Theorem 4.1, where the initial positions are randomly chosen. On the other hand, Fig. 8(a) shows that all agents converge to a framework equivalent but not congruent to the target framework under the conventional control law (Chen et al., 2017; Sun, Mou, Anderson & Cao, 2016), where the initial positions are the same as those in Fig. 8(b). It is shown in Fig. 8(c) and Fig. 8(d) that if the coefficient K is not sufficiently large then the initial agents may converge to an undesired signed framework even though all conditions are the same as in Fig. 8(b) except for K.

5.2. Under the unicycle model-based system

As shown in Fig. 9, we can observe that, under the control law (17), the unicycle agents converge to a signed framework in the same attraction region as the system (12).

6. Conclusion

This paper has developed sign rigidity theory which includes the three sub-concepts of sign rigidity, global sign rigidity, and infinitesimal sign rigidity. The sign rigidity theory can contribute to specification of formation shapes and arrangements of agents with distance- and signed area-constraints. In particular, due to the signed areas in a formation specification, we can eliminate certain formation specification ambiguities. Moreover, this paper proposes the signed Henneberg construction to achieve a globally signed rigid formation without any formation specification ambiguities. As an application of the sign rigidity theory, this paper explores the formation specification control applied to two types of agents. For both systems of single integrator models and unicycle models, if a target signed formation is generated by the operation of signed Henneberg construction with sufficiently large K, then almost global convergence is achieved in $\mathbb{R}^2$ without any formation specification ambiguities. In particular, the formation control systems are distributed and do not require a global (common) coordinate system and coordinate frame orientation information of neighbor agents.

We remark further research directions as follows. The first further study would be to develop the sign rigidity theory in 3-D space, and to apply the theory to formation specification control in 3-D space. We expect that the concept of signed volume can be employed instead of the signed area in the same sense as the work (Kwon et al., 2019). The second study would be to explore how large K needs to be for almost global convergence. The third study would be to study network localization problems based on the result of almost global formation stabilization studied in this paper. Since it is well known that the two problems of formation control and network localization are regarded as a duality (Ahn, 2020), we expect that our work can contribute to solving multi-agent localization problems. We also expect that our work can be applied to flocking control with double-integrator dynamics (Deghat et al., 2015; Sun, Anderson, Deghat & Ahn, 2017; Sun, Mou, Deghat & Anderson, 2016) and formation maneuvering (Cai & de Queiroz, 2015; Chen, de Marina, & Cao, 2021; Mehdifar et al., 2018).

Appendix A. Proof of Proposition 4.2

To prove Proposition 4.2, we need to define new notations and derive a useful lemma. We first separate a signed framework $(\mathcal{G}, \mathcal{S}, p)$ into ζ_s sub-frameworks in accordance with sequences of the signed Henneberg construction as follows: Given a signed framework $(\mathcal{G}, \mathcal{S}, p)$, we partition $(\mathcal{G}, \mathcal{S}, p)$ into $(\mathcal{G}_i, \mathcal{S}_i, c_i)$, $i \in \{1, 2, \ldots, \zeta_s\}$ with three agents, where $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ denotes a sub-graph and $c_i = [p_j^\top, p_k^\top, p_l^\top]^\top \in \mathbb{R}^6$ for $j, k, l \in \mathcal{V}_i$ denotes a realization for a sub-framework. It is assumed that $\mathcal{E} = \bigcup_{i=1}^{\zeta_s} \mathcal{E}_i$ and $\mathcal{E}_i \cap \mathcal{E}_{i'} = \emptyset$ for $i, i' \in \{1, 2, \ldots, \zeta_s\}, i \neq i'$ with $|\mathcal{E}_i| = 3$ if $i = 1$, otherwise $|\mathcal{E}_i| = 2$; $\mathcal{S} = \bigcup_{i=1}^{\zeta_s} \mathcal{S}_i$ and $\mathcal{S}_i \cap \mathcal{S}_{i'} = \emptyset$ for $i, i' \in \{1, 2, \ldots, \zeta_s\}, i \neq i'$ with $|\mathcal{S}_i| = 1$. It then follows that $\mathcal{V} = \bigcup_{i=1}^{\zeta_s} \mathcal{V}_i$ with $|\mathcal{V}_i| = 3$, and if c_i adjoins $c_{i'}$ for $i, i' \in \{1, 2, \ldots, \zeta_s\}, i \neq i'$ then $\mathcal{V}_i \cap \mathcal{V}_{i'} \neq \emptyset$, otherwise $\mathcal{V}_i \cap \mathcal{V}_{i'} = \emptyset$. For example, see Fig. A.1 and Fig. A.2.

We next consider the following Hessian matrix of ϕ which is the same as the negative Jacobian matrix of (12):

$$\mathbf{H}_p = \frac{\partial^2 \phi}{\partial p^2} \in \mathbb{R}^{2n \times 2n} \tag{A.1}$$

According to the split sub-frameworks, we define several functions. The vectors constituting distance constraints and their desired values are defined as

$$f_{d_i}^s(c_i) = \frac{1}{2}\left[\cdots, \|p_k - p_j\|^2, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{E}_i|}, \tag{A.2}$$

$$f_{d_i}^{s*} = \frac{1}{2}\left[\cdots, \|p_k - p_j\|^{*2}, \ldots\right]^\top \in \mathbb{R}^{|\mathcal{E}_i|}, \tag{A.3}$$

for $i \in \{1, 2, \ldots, \zeta_s\}$ and $(j, k) \in \mathcal{E}_i$. A signed area constraint and its desired value for a sub-framework is denoted by

$$f_{a_i}^s(c_i) = KA_{jkl} \in \mathbb{R}, \tag{A.4}$$

$$f_{a_i}^{s*} = KA_{jkl}^* \in \mathbb{R}, \tag{A.5}$$

where $i \in \{1, 2, \ldots, \zeta_s\}$ and $(j, k, l) \in \mathcal{S}_i$. Then, the formation specification error for each sub-framework is defined as

$$e_i(c_i) = \left[f_{d_i}^{s\top}(c_i) \quad f_{a_i}^s(c_i)\right]^\top - \left[f_{d_i}^{s*\top} \quad f_{a_i}^{s*}\right]^\top. \tag{A.6}$$

Similarly to (A.1), the Hessian matrix of a potential function $\phi_i = \frac{1}{2}\|e_i(c_i)\|^2$, $i \in \{1, 2, \ldots, \zeta_s\}$ for a sub-framework is given as

$$\mathbf{H}_{c_i} = \frac{\partial^2 \phi_i}{\partial c_i^2} \in \mathbb{R}^{6 \times 6}. \tag{A.7}$$

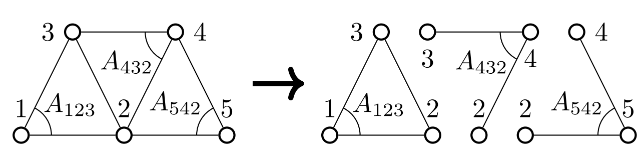
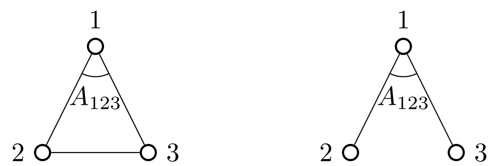


Fig. A.1. Example of separating a signed framework, where $\mathcal{V}_1 = \{1, 2, 3\}$, $\mathcal{V}_2 = \{2, 3, 4\}$, $\mathcal{V}_3 = \{2, 4, 5\}$, $\mathcal{E}_1 = \{(1, 2), (1, 3), (2, 3)\}$, $\mathcal{E}_2 = \{(2, 4), (3, 4)\}$, $\mathcal{E}_3 = \{(2, 5), (4, 5)\}$, $\mathcal{S}_1 = \{(1, 2, 3)\}$, $\mathcal{S}_1 = \{(4, 3, 2)\}$, and $\mathcal{S}_1 = \{(5, 4, 2)\}$.



(a) $(\mathcal{G}_1, \mathcal{S}_1, c_1)$ (b) $(\mathcal{G}_i, \mathcal{S}_i, c_i)$ for $i \in \{2, 3, \cdots, \zeta_s\}$

Fig. A.2. Example of sub-framework $(\mathcal{G}_i, \mathcal{S}_i, c_i)$ for $i \in \{1, 2, \ldots, \zeta_s\}$, where the distances are denoted by solid lines.

The signed rigidity matrix for a sub-framework is defined as

$$\overline{R}^s_{d_i} = \frac{\partial f^s_i}{\partial c_i} \in \mathbb{R}^{(|\mathcal{E}_i|+1)\times 6}, i \in \{1, 2, \ldots, \zeta_s\}, \tag{A.8}$$

where $f^s_i = \left[f^{s\top}_{d_i}(c_i), f^s_{a_i}(c_i)\right]^\top \in \mathbb{R}^{|\mathcal{E}_i|+1}$. The definition of $\overline{R}^s_{d_i}$ is slightly different from R^s_d in (5) due to the coefficient K in the signed area constraint (A.4); however, the matrix property such as rank and null space remains satisfied. Then, one can express the Hessian matrix of ϕ_i in the following form:

$$\mathbf{H}_{c_i} = \overline{R}^{s\top}_{d_i}\overline{R}^s_{d_i} + E_d \otimes I_2 + E_s \otimes J. \tag{A.9}$$

where E_d and E_s denote 3×3 matrices composed of errors associated with distances and signed areas, respectively. For example, in Fig. A.2, E_d and E_s are given by

$$E_d = \begin{cases} \begin{bmatrix} e_{d_{12}} + e_{d_{13}} & -e_{d_{12}} & -e_{d_{13}} \\ -e_{d_{12}} & e_{d_{23}} + e_{d_{12}} & -e_{d_{23}} \\ -e_{d_{31}} & -e_{d_{23}} & e_{d_{31}} + e_{d_{23}} \end{bmatrix} & \text{for } \mathcal{E}_1, \\ \begin{bmatrix} e_{d_{12}} + e_{d_{13}} & -e_{d_{12}} & -e_{d_{13}} \\ -e_{d_{12}} & e_{d_{12}} & 0 \\ -e_{d_{31}} & 0 & e_{d_{31}} \end{bmatrix} & \text{for } \mathcal{E}_i, i \geq 2, \end{cases} \tag{A.10}$$

$$E_s = \begin{bmatrix} 0 & \frac{K}{2}e_{a_{123}} & -\frac{K}{2}e_{a_{123}} \\ -\frac{K}{2}e_{a_{123}} & 0 & \frac{K}{2}e_{a_{123}} \\ \frac{K}{2}e_{a_{123}} & -\frac{K}{2}e_{a_{123}} & 0 \end{bmatrix}, \tag{A.11}$$

where $e_{d_{jk}}$ is defined as $e_{d_{jk}} = \frac{1}{2}\|p_k - p_j\|^2 - \frac{1}{2}\|p_k - p_j\|^{*2}$ for $(j, k) \in \mathcal{E}_i, i \in \{1, 2, \ldots, \zeta_s\}$, and $e_{a_{jkl}}$ is defined as $e_{a_{jkl}} = K(A_{jkl} - A^*_{jkl})$ for $(j, k, l) \in \mathcal{S}_i, i \in \{1, 2, \ldots, \zeta_s\}$; for more details, refer to Appendix B.

We here introduce a permutation matrix $P \in \mathbb{R}^{6\times 6}$ such that

$$\overline{R}^s_{d_i}P = \begin{bmatrix} \bar{R}_x & \bar{R}_y \end{bmatrix} = R^P_i, \tag{A.12}$$

where $\bar{R}_u \in \mathbb{R}^{(|\mathcal{E}_i|+1)\times 3}$ for $u = x, y$ is a matrix whose columns are reordered in accordance with coordinate u in the matrix $\overline{R}^s_{d_i}$. With this permutation matrix, we have the following observation:

$$\begin{aligned} \mathbf{H}^P_{c_i} =& P^\top \mathbf{H}_{c_i} P \\ =& R^{P\top}_i R^P_i + I_2 \otimes E_d + J \otimes E_s \\ =& \begin{bmatrix} \bar{R}^\top_x \bar{R}_x + E_d & \bar{R}^\top_x \bar{R}_y + E_s \\ \bar{R}^\top_y \bar{R}_x - E_s & \bar{R}^\top_y \bar{R}_y + E_d \end{bmatrix}, \end{aligned} \tag{A.13}$$

where we have used the fact that $PP^\top = I_6$.

A useful lemma to prove Proposition 4.2 is given as follows.

Lemma A.1. *Assume that c_j is a regular point*[7] *of f^s_j for $j \in \{1, 2, \ldots, \zeta_s\}$. Then, there exists a derivative map $D\eta_i$ of a map $\eta_i(c_i)|_{\mathcal{G}_j}$ for $c_i \in \mathcal{B}_{c_j}$ close to $(\mathcal{G}_j, \mathcal{S}_j, c_j)$ such that*

$$D\eta_i(v_i)|_{\mathcal{G}_j} = \begin{cases} v_i \in \mathbb{R}^{2|\mathcal{V}_i|} & \textit{if } i = j, \\ v_i \in T_{\eta_i(c_i)} f^{s^{-1}}_j(f^s_j(c_j)) & \textit{otherwise}, \end{cases} \tag{A.14}$$

where $\mathcal{B}_{c_j}$ denotes a neighborhood of c_j, and $T_x\mathcal{M}$ denotes the tangent space to a manifold $\mathcal{M}$ at a point $x \in \mathcal{M}$.

Proof. Let us define a function $\eta_i(c_i)|_{\mathcal{G}_j}$ near $(\mathcal{G}_j, \mathcal{S}_j, c_j)$ as $\eta_i(c_i)|_{\mathcal{G}_j} = c_i$ for $c_i \in \mathcal{B}_{c_j}$. This equation is equivalent to

$$\eta_i(c_i)|_{\mathcal{G}_j} = \begin{cases} c_i \in \mathbb{R}^{2|\mathcal{V}_i|} & \text{if } i = j, \\ c_i \in \mathcal{B}_{c_j} \subseteq \mathbb{R}^{2|\mathcal{V}_i|} & \text{otherwise}, \end{cases} \tag{A.15}$$

where, in terms of the case $i = j$, a sub-framework c_i completely belongs to $\mathcal{B}_{c_i}$ since $\mathcal{B}_{c_i}$ is a ball centered at c_i, which means that c_i is an interior point of $\mathcal{B}_{c_i}$.

Based on Asimow and Roth (1978, Proposition 2), since c_j is a regular point of f^s_j, there exists a neighborhood $\mathcal{B}_{c_j}$ such that $f^{s^{-1}}_j(f^s_j(c_j)) \cap \mathcal{B}_{c_j}$ is a $(2|\mathcal{V}_j| - r^m_{f^s_j})$-dimensional smooth manifold, where $r^m_{f^s_j} = \max\{\text{rank}(\frac{\partial f^s_j}{\partial c_j})|c_j \in \mathbb{R}^{2|\mathcal{V}_j|}\}$. Then, it follows from (A.15) that, for $i \neq j$, there exists $\eta_i(c_i)|_{\mathcal{G}_j}$ such that

$$\eta_i(c_i)|_{\mathcal{G}_j} = c_i \in f^{s^{-1}}_j(f^s_j(c_j)). \tag{A.16}$$

Thus, we can have

$$\eta_i(c_i)|_{\mathcal{G}_j} = \begin{cases} c_i \in \mathbb{R}^{2|\mathcal{V}_i|} & \text{if } i = j, \\ c_i \in f^{s^{-1}}_j(f^s_j(c_j)) & \text{otherwise}. \end{cases} \tag{A.17}$$

We can also observe a derivative map $D\eta_i$ for $i \neq j$ such that $D\eta_i(v_i)|_{\mathcal{G}_j} = v_i \in T_{\eta_i(c_i)} f^{s^{-1}}_j(f^s_j(c_j))$. We then have

$$D\eta_i(v_i)|_{\mathcal{G}_j} = \begin{cases} v_i \in \mathbb{R}^{2|\mathcal{V}_i|} & \text{if } i = j, \\ v_i \in T_{\eta_i(c_i)} f^{s^{-1}}_j(f^s_j(c_j)) & \text{otherwise}. \end{cases} \tag{A.18}$$

Therefore, the proof is completed. ■

The condition that $c_i, i \in \{1, 2, \ldots, \zeta_s\}$ is a regular point of f^s_i implies that each sub-framework $(\mathcal{G}_i, \mathcal{S}_i, c_i)$ is infinitesimally sign rigid from Lemma 3.1 and Theorem 3.1. Moreover, this implies that the tangent vectors of $f^{s^{-1}}_i(f^s_i(c_i))$ for $i \in \{1, 2, \ldots, \zeta_s\}$ are the motions of $(\mathcal{G}_i, \mathcal{S}_i, c_i)$ in $SE(2)$ (Asimow & Roth, 1978, Proposition 2). We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. This proof is done by contradiction. W assume that $(\mathcal{G}, p^*)$ with a stable equilibrium point p^* is not strongly distance rigid. Then, there exists a realization $c^\dagger_i$ for a sub-framework such that three agents at $c^\dagger_i$ are collinear. Then, without loss of generality, we can assume that $\bar{R}_y = 0$ at $c^\dagger_i$, which implies from (A.13) that

$$\mathbf{H}^P_{c_i}(c^\dagger_i) = \begin{bmatrix} \bar{R}^\top_x(c^\dagger_i)\bar{R}_x(c^\dagger_i) + E_d(c^\dagger_i) & E_s(c^\dagger_i) \\ -E_s(c^\dagger_i) & E_d(c^\dagger_i) \end{bmatrix}. \tag{A.19}$$

We here use the fact that $E_d(c^\dagger_i)$ has at least one negative eigenvalue as studied in Sun (2018, Lemma 3.2). With a nonzero vector defined as $\bar{v} = [\bar{v}^\top_1, \bar{v}^\top_2]^\top$, where $\bar{v}_i \in \mathbb{R}^3$ for $i = 1, 2$ with $\bar{v}_1 = 0$, we can observe that $\bar{v}^\top \mathbf{H}^P_{c_i}(c^\dagger_i)\bar{v} = \bar{v}^\top_2 E_d(c^\dagger_i)\bar{v}_2$, which implies that there exists a nonzero vector $\bar{v}$ such that $\bar{v}^\top \mathbf{H}^P_{c_i}(c^\dagger_i)\bar{v} < 0$. Thus, we have the fact that there exists a negative eigenvalue of $\mathbf{H}_{c_i}(c^\dagger_i)$ since $\mathbf{H}_{c_i}$ has the same spectrum as $\mathbf{H}^P_{c_i}$.

Since $\phi = \sum^{\zeta_s}_{i=1}\phi_i$, it holds that $\mathbf{H}_p = \sum^{\zeta_s}_{i=1}\widehat{\mathbf{H}}_{c_i}$, where $\widehat{\mathbf{H}}_{c_i} = \frac{\partial^2\phi_i}{\partial p^2} \in \mathbb{R}^{2n\times 2n}$. We here use the observation that $\widehat{\mathbf{H}}_{c_i}$ can be obtained from $\mathbf{H}_{c_i}$ by adding new zero-rows/columns into $\mathbf{H}_{c_i}$. Thus, there exists a nonzero vector $\hat{v}_i \in \mathbb{R}^{2n}$ such that $\hat{v}^\top_i \widehat{\mathbf{H}}_{c_i}(p^*)\hat{v}_i < 0$ if $\mathbf{H}_{c_i}$ has at least one negative eigenvalue at $c^\dagger_i$. It is noted that $\mathbf{H}_p$ is not a block diagonal matrix composed of

[7] (See Asimow & Roth, 1978) A realization c_i is called a *regular point* of f^s_i if $\text{rank}(\frac{\partial f^s_i}{\partial c_i}) = r^m_{f^s_i}$, where $r^m_{f^s_i} = \max\{\text{rank}(\frac{\partial f^s_i}{\partial c_i})|c_i \in \mathbb{R}^{2|\mathcal{V}_i|}\}$.

$\mathbf{H}_{c_i}, i \in \{1, \ldots, \zeta_s\}$, i.e., $\mathbf{H}_p \neq \mathrm{diag}(\mathbf{H}_{c_1}, \mathbf{H}_{c_2}, \ldots, \mathbf{H}_{c_{\zeta_s}})$, since a sub-framework $(\mathcal{G}_i, \mathcal{S}_i, c_i), i \in \{1, 2, \ldots, \zeta_s\}$ shares at least one agent with an adjacent sub-framework $(\mathcal{G}_j, \mathcal{S}_j, c_j), j \in \{1, \ldots, \zeta_s\} \setminus \{i\}$; refer to Fig. A.1. Therefore, we cannot directly conclude that there exists a negative eigenvalue of $\mathbf{H}_p(p^*)$ from the result that $\mathbf{H}_{c_i}(c_i^\dagger)$ has at least one negative eigenvalue. Nevertheless, we can approach the conclusion from the fact that $\mathbf{H}_{c_i}(c_i^\dagger)$ (equivalently, $\widehat{\mathbf{H}}_{c_i}(p^*)$) has a negative eigenvalue(s) since $(\mathcal{G}, \mathcal{S}, p)$ has the signed Henneberg construction. In the following, we show that $\mathbf{H}_p(p^*)$ has at least one negative eigenvalue with the fact that $\widehat{\mathbf{H}}_{c_i}(p^*)$ has at least a negative eigenvalue.

Considering a sub-framework $(\mathcal{G}_1, \mathcal{S}_1, c_1)$, we derive an explicit formula of the sign rigidity matrix as given in Eq. (B.3). The Eq. (B.3) has a similar form as the control system studied in Anderson et al. (2017) for the 3-agent case. In the same way as Anderson et al. (2017, Theorems 1, 2 and 3), we can have that, for sufficiently large K, $e_1 = 0$ is a unique solution to $\overline{R}_{d_1}^{s\top} e_1$ and further $\overline{R}_{d_1}^{s}$ is of maximum row rank. In the same sense, we can have $\overline{R}_{d_j}^{s}, j \in \{2, 3, \ldots, \zeta_s\}$ being of maximum row rank for sufficiently large K. These mean that $c_j, j \in \{1, 2, \ldots, \zeta_s\}$ is a regular point of f_j^s for sufficiently large K. It then follows from Lemma A.1 that, since $\mathrm{span}(L_p) \subset \mathrm{null}\left(\widehat{\mathbf{H}}_{c_i}\right)$ (Chen et al., 2017; Field, 1980), there exists a vector $D\eta_i(\hat{v}_i)|_{\mathcal{G}_j}$ such that

$$\left(D\eta_i(\hat{v}_i)|_{\mathcal{G}_j}\right)^\top \widehat{\mathbf{H}}_{c_i}(p^*) D\eta_i(\hat{v}_i)|_{\mathcal{G}_j} = \begin{cases} \omega < 0 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \tag{A.20}$$

With reference to the equality $\mathbf{H}_p = \sum_{i=1}^{\zeta_s} \widehat{\mathbf{H}}_{c_i}$, it follows from (A.20) that $\left(D\eta_i(\hat{v}_i)|_{\mathcal{G}_j}\right)^\top \mathbf{H}_p(p^*) D\eta_i(\hat{v}_i)|_{\mathcal{G}_j} = \omega < 0$. Therefore, we have shown that, for sufficiently large K, if $(\mathcal{G}, p^*)$ is not strongly distance rigid then there exists a negative eigenvalue of $\mathbf{H}_p(p^*)$, i.e., $p = p^*$ is unstable. However, this is a contradiction. Therefore, $(\mathcal{G}, p^*)$ is strongly distance rigid. ■

Appendix B. Calculation example of (A.9)

The following calculations are based on the example in Fig. A.2(a) in Appendix A. The Hessian matrix is given by

$$\mathbf{H}_{c_1} = \frac{\partial^2 \phi_1}{\partial c_1^2} = \frac{\partial}{\partial c_1}(\overline{R}_{d_1}^{s\top} e_1) \in \mathbb{R}^{6\times 6}, \tag{B.1}$$

where $e_1(c_1) = \left[f_{d_1}^{s\top}(c_1), f_{a_1}^{s}(c_1)\right]^\top - \left[f_{d_1}^{s*\top}, f_{a_1}^{s*}\right]^\top$, $\phi_1 = \frac{1}{2}\|e_1(c_1)\|^2$ and $c_1 = [p_1^\top, p_2^\top, p_3^\top]^\top$. The signed rigidity matrix is given as

$$\overline{R}_{d_1}^{s} = \frac{\partial f_1^s}{\partial c_1} = \begin{bmatrix} z_{12}^\top & -z_{12}^\top & 0 \\ z_{13}^\top & 0 & -z_{13}^\top \\ 0 & z_{23}^\top & -z_{23}^\top \\ \frac{K}{2}(z_{12}^\top J + z_{13}^\top J^\top) & -\frac{K}{2} z_{13}^\top J^\top & -\frac{K}{2} z_{12}^\top J \end{bmatrix} \tag{B.2}$$

where $z_{jk} = p_k - p_j$ for $(j,k) \in \mathcal{E}_1$ and $f_1^s = \left[f_{d_1}^{s\top}(c_1), f_{a_1}^{s}(c_1)\right]^\top \in \mathbb{R}^{|\mathcal{E}_1|+1}$. Then, $\overline{R}_{d_1}^{s\top} e_1$ yields

$$\overline{R}_{d_1}^{s\top} e_1 = \begin{bmatrix} e_{d_{12}} z_{12} + e_{d_{13}} z_{13} + e_{a_{123}} \frac{K}{2}(J^\top z_{12} + J z_{13}) \\ -e_{d_{12}} z_{12} + e_{d_{23}} z_{23} - e_{a_{123}} \frac{K}{2}(J z_{13}) \\ -e_{d_{13}} z_{13} - e_{d_{23}} z_{23} - e_{a_{123}} \frac{K}{2}(J^\top z_{12}) \end{bmatrix} \tag{B.3}$$

where $e_{d_{jk}} = \frac{1}{2}\|z_{jk}\|^2 - \frac{1}{2}\|z_{jk}\|^{*2}$ for $(j,k) \in \mathcal{E}_1$ and $e_{a_{jkl}} = K(A_{jkl} - A_{jkl}^*)$ for $(j,k,l) \in \mathcal{S}_1$. Based on the above notations, we finally have

$$\begin{aligned} \mathbf{H}_{c_1} &= \frac{\partial}{\partial c_1}(\overline{R}_{d_1}^{s\top} e_1) \in \mathbb{R}^{6\times 6}, \\ &= E_d \otimes I_2 + E_s \otimes J + \begin{bmatrix} C_{p_1} & C_{p_2} & C_{p_3} \end{bmatrix}, \end{aligned} \tag{B.4}$$

where

$$C_{p_1} = \begin{bmatrix} z_{12} z_{12}^\top + z_{13} z_{13}^\top + \frac{K^2}{4}(J^\top z_{12} + J z_{13})(J^\top z_{12} + J z_{13})^\top \\ -z_{12} z_{12}^\top - \frac{K^2}{4}(J z_{13})(J^\top z_{12} + J z_{13})^\top \\ -z_{13} z_{13}^\top - \frac{K^2}{4}(J^\top z_{12})(J^\top z_{12} + J z_{13})^\top \end{bmatrix},$$

$$C_{p_2} = \begin{bmatrix} -z_{12} z_{12}^\top - \frac{K^2}{4}(J^\top z_{12} + J z_{13})(J z_{13})^\top \\ z_{12} z_{12}^\top + z_{23} z_{23}^\top + \frac{K^2}{4}(J z_{13})(J z_{13})^\top \\ -z_{23} z_{23}^\top + \frac{K^2}{4}(J^\top z_{12})(J z_{13})^\top \end{bmatrix},$$

$$C_{p_3} = \begin{bmatrix} -z_{13} z_{13}^\top - \frac{K^2}{4}(J^\top z_{12} + J z_{13})(J^\top z_{12})^\top \\ -z_{23} z_{23}^\top + \frac{K^2}{4}(J z_{13})(J^\top z_{12})^\top \\ z_{13} z_{13}^\top + z_{23} z_{23}^\top + \frac{K^2}{4}(J^\top z_{12})(J^\top z_{12})^\top \end{bmatrix},$$

and it holds that $\begin{bmatrix} C_{p_1} & C_{p_2} & C_{p_3} \end{bmatrix} = \overline{R}_{d_1}^{s\top} \overline{R}_{d_1}^{s}$.

Appendix C. Proof of Lemma 4.1

The following statements follow the notations defined in Appendix A. Let us first consider a realization $p^\dagger$ at which there is an incorrect sign of a signed area; for example, an incorrect sign of a signed area occurs in case of $A_{jkl} = 0$ or $A_{jkl} < 0$ with $A_{jkl}^* > 0$ for $(j,k,l) \in \mathcal{S}$. Then, we can observe that, at $p^\dagger$, $A_{jkl} - A_{jkl}^* \neq 0$, $(j,k,l) \in \mathcal{S}_i$ for a sub-framework $(\mathcal{G}_i, \mathcal{S}_i, c_i)$, $i \in \{1, 2, \ldots, \zeta_s\}$. With this fact, we next show that, for sufficiently large K, there exists a negative eigenvalue of $\mathbf{H}_p$ at $p^\dagger$, that is, $p = p^\dagger$ is unstable for sufficiently large K.

This argument is proved by contradiction. We first assume that $p = p^\dagger$ is stable for sufficiently large K. Let us consider Eq. (A.9) rewritten with respect to the definition $\overline{R}_{d_i}^{s} = \begin{bmatrix} R_{d_i}^\top & K R_{s_i}^\top \end{bmatrix}^\top$, where $R_{d_i} = \frac{\partial f_{d_i}^s}{\partial c_i}$ and $R_{s_i} = \frac{\partial}{\partial c_i} A_{jkl}$ for $(j,k,l) \in \mathcal{S}_i, i \in \{1, 2, \ldots, \zeta_s\}$, as follows:

$$\begin{aligned} \mathbf{H}_{c_i} &= \overline{R}_{d_i}^{s\top} \overline{R}_{d_i}^{s} + E_d \otimes I_2 + E_s \otimes J \\ &= R_{d_i}^\top R_{d_i} + K^2 R_{s_i}^\top R_{s_i} + E_d \otimes I_2 + E_s \otimes J. \end{aligned} \tag{C.1}$$

Then, we can choose a nonzero vector $x \in \mathbb{R}^6$ to have $x^\top \mathbf{H}_{c_i} x < 0$. To show this, we define the vector x as $x = [x_1^\top, \epsilon x_2^\top, 0]^\top \in \mathbb{R}^6$, where $x_1 \in \mathbb{R}^2$ is a unit vector orthogonal to $J^\top(p_l - p_k)$ for $(j,k,l) \in \mathcal{S}_i$ and $i \in \{1, 2, \ldots, \zeta_s\}$, $x_2 \in \mathbb{R}^2$ is a nonzero vector, and ϵ is a small positive constant. With the definition of x, we have

$$\begin{aligned} & x^\top(K^2 R_{s_i}^\top R_{s_i} + E_s \otimes J)x \\ &= K^2 \underbrace{\epsilon^2 x_2^\top J^\top (p_l - p_j)(p_l - p_j)^\top J x_2}_{=x^\top R_{s_i}^\top R_{s_i} x} + \underbrace{K \epsilon x_1^\top J x_2 e_{a_{jkl}}}_{=x^\top (E_s \otimes J) x} \\ &= K^2(\epsilon x_1^\top J x_2 (A_{jkl} - A_{jkl}^*) + O(\epsilon^2)), \end{aligned} \tag{C.2}$$

where $e_{a_{jkl}} = K(A_{jkl} - A_{jkl}^*)$ and we have used the fact that $R_{s_i} = \frac{\partial A_{jkl}}{\partial c_i} = \left[(p_l - p_k)^\top J, \quad (p_l - p_j)^\top J, \quad (p_k - p_j)^\top J\right]$ for $(j,k,l) \in \mathcal{S}_i, i \in \{1, 2, \ldots, \zeta_s\}$. Since the two terms $R_{d_i}^\top R_{d_i}$ and $E_d \otimes I_2$ in (C.1) are independent of K, we can have the following form from (C.1) and (C.2) for sufficiently large K:

$$x^\top \mathbf{H}_{c_i} x = K^2 \left(\epsilon x_1^\top J x_2 (A_{jkl} - A_{jkl}^*) + O(\epsilon^2) + O(\bar{\epsilon}^2)\right),$$

where $\bar{\epsilon} = 1/K$, which implies that we can choose x such that $x_1^\top J x_2 (A_{jkl} - A_{jkl}^*) < 0$ and further $x^\top \mathbf{H}_{c_i} x < 0$ at $p^\dagger$. Thus, for sufficiently large K, there exists a negative eigenvalue of $\mathbf{H}_{c_i}$ in case of an incorrect sign for $(j,k,l) \in \mathcal{S}_i$. Referring to the proof of Proposition 4.2 in Appendix A, it holds that $\mathbf{H}_p = \sum_{i=1}^{\zeta_s} \widehat{\mathbf{H}}_{c_i}$ and there exists a nonzero vector $\hat{x} \in \mathbb{R}^{2n}$ obtained from x such that $\hat{x}^\top \widehat{\mathbf{H}}_{c_i}\hat{x} < 0$ for $i \in \{1, 2, \ldots, \zeta_s\}$ at $p^\dagger$, which leads to $\left(D\eta_i(\hat{x})|_{\mathcal{G}_j}\right)^\top \mathbf{H}_p(p^\dagger) D\eta_i(\hat{x})|_{\mathcal{G}_j} < 0$ from Lemma A.1. However, this contradicts the assumption that $p = p^\dagger$ is stable for sufficiently large K. Therefore, under the system (12), $p = p^\dagger$ is unstable for sufficiently large K. ■

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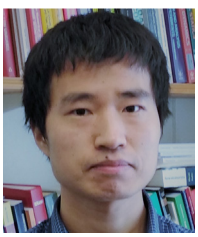

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