Received: 28 August 2021 | Revised: 14 February 2022 | Accepted: 20 June 2022

DOI: 10.1002/rnc.6267

SHORT COMMUNICATION

WILEY

Free-will arbitrary time consensus protocols with diffusive coupling

Quoc Van Tran[1] | **Minh Hoang Trinh**[2] | **Nam Hoai Nguyen**[2] | **Hyo-Sung Ahn**[3]

[1]Department of Mechatronics, School of Mechanical Engineering, Hanoi University of Science and Technology (HUST), Hanoi, Vietnam

[2]Department of Automation Engineering, School of Electrical and Electronic Engineering, Hanoi University of Science and Technology (HUST), Hanoi, Vietnam

[3]School of Mechanical Engineering, Gwangju Institute of Science and Technology (GIST), Gwangju, Republic of Korea

Correspondence
Hyo-Sung Ahn, School of Mechanical Engineering, Gwangju Institute of Science and Technology (GIST), Gwangju 61005, Republic of Korea.
Email: hyosung@gist.ac.kr

Funding information
National Research Foundation of Korea, Grant/Award Number: 2022R1A2B5B03001459; Ministry of Science and Technology in Vietnam, Grant/Award Number: KC-4.0-34/19-25

Abstract

In this article, free-will arbitrary time consensus protocols are proposed for multi-agent systems with single- and double- (possibly higher-order) integrator dynamics, respectively, and with (possibly switching) connected interaction graphs and bounded matched disturbances. Under the proposed consensus laws, an average consensus is achieved in a free-will arbitrary prespecified time regardless of the initial conditions or any other design parameters. Further, the proposed consensus laws for the case with no disturbances are smooth, and they are distributed in the sense that information is only communicated locally between neighboring agents. Finally, simulation results are also provided to illustrate the theoretical results and, an application to arbitrary prespecified time formation control of mobile agents is also presented.

KEYWORDS

disturbance, high-order integrator, multi-agent systems, prespecified time consensus, switching graphs

1 | INTRODUCTION

Achieving a consensus on some local decision states is a crucial task in distributed control and estimation over networked systems.[1-4] In this context, each agent in the system holds a local state, obtains the states of other agents via inter-agent measurements or communication, and updates its state in time such that its local disagreement with its neighboring agents vanishes.

Many problems involving multi-agent systems (MASs), including orientation localization,[5,6] orientation stabilization,[7,8] and coordination control,[9-13] require the agents' states to reach a consensus within a finite time for high precision performance. Thus, finite-time control and estimation in MASs have attracted tremendous research attention in recent years.[5,14-19] However, an upper bound, namely t_f, of the convergence time in finite-time (FT) consensus in general depends on the initial condition and other design parameters,[5,14-16,19] which means that t_f cannot be chosen freely. Finite-time consensus has been investigated for MASs with second-order nonlinear switched dynamics,[17] with leader-following and strongly connected graphs,[19] and with input saturation and disturbance.[18] In Reference 5, finite-time estimation of the agents' orientation matrices and finite-time bearing-only formation control were studied for systems with rigid graphs. Discontinuous consensus protocols based on both the in- and out-Laplacian matrices of a directed graph were proposed in Reference 15 to address finite-time opinion dynamics and distributed multi-agent optimization.

Fixed-time (FxT) consensus schemes have been proposed in References 20-24, whose the upper bound of the convergence time is independent of the initial condition. Nevertheless, the bound of the settling time in fixed-time control is still dependent on the design parameters, and hence cannot be assigned arbitrarily. Moreover, such estimates of the bound of the convergence time in fixed-time (and finite-time) consensus are very conservative, as shown in References 22 and 23. Fixed-time consensus protocols for MASs with switching graph topologies and general directed graphs were investigated in References 23 and 21, respectively. The authors in Reference 22 addressed fixed-time consensus for a class of heterogeneous MASs with first-order dynamics. The work[6] studied fixed-time orientation estimation and network localization in the two-dimensional plane, and fixed-time network localization laws using bearing measurements were proposed in Reference 25. Consensus laws using a norm-normalized signum function, proposed in Reference 24, achieve a fixed-time synchronized consensus, in which all the state vector's elements converge to the origin at the same time. Consensus laws with prespecified convergence time using an auxiliary time-varying gain were proposed in Reference 12. An extension of Reference 12 to prespecified time bearing-only formation control was given in Reference 26. Recently, free-will arbitrary time (FWAT) consensus protocols, built upon the results in Reference 27, have been presented in References 28 and 29. In FWAT consensus, the settling time is bounded by a preset finite time t_f, which does not depend on the initial condition or any design parameter. The settling time bound t_f is explicitly available in the designed consensus laws and can be prespecified arbitrarily.[28,29] Furthermore, the design and convergence analysis of FWAT consensus laws[28,29] are simpler than those in References 12 and 26.

For second-order dynamics, finite/fixed-time consensus laws have been proposed based on sliding mode control.[18,24,30,31] These control schemes, however, become non-smooth or may suffer from chartering effect due to the use of the signum function. Another issue, arising when the finite-/fixed-time consensus protocols for first-order dynamics are directly extended to the second-order counterpart, is the singularity in the control law.[32,33] In Reference 33, fixed-time consensus schemes for double-integrator agents were presented, to which a sinusoid function is introduced to avoid such a singularity. A consensus law with complex switching sliding modes was investigated in Reference 24 to eliminate the control singularity. The authors in Reference 34 explored a fixed-time consensus method for high-order integrator agents with leader-following graphs, in which the graph of the followers is undirected and connected, based on fixed-time distributed estimation of the leader's state. Furthermore, existing works in prespecified time consensus[12,28,29] have often been proposed for first-order integrator dynamics. Consequently, this work aims to investigate FWAT consensus schemes for systems of single- and double-integrator (possibly higher-order integrator) modeled agents, respectively, possibly in the presence of switching graph topologies or matched disturbances.

The specific contributions of this article are as follows.

- First, a FWAT consensus law for systems of single-integrator agents and with (possibly switching) connected interaction graphs is proposed. In the presence of bounded disturbances, based on an integral sliding mode method,[18,28,30,35] a discontinuous term is added to the consensus law to reject the disturbance. The proposed consensus law in addition overcomes the technical issue associated with the FWAT consensus law in Reference 28, as will be clarified in the main text.
- Second, compared with References 12,28,29, and 26 that propose prespecified-time consensus laws for only first-order integrator dynamics, this work proposes FWAT consensus schemes for MASs with p-order-integrator dynamics ($p \geq 2$). In particular, a tracking control scheme is presented to reduce the p-order system to the first-order counterpart in a free-will arbitrary time, and subsequently, a FWAT consensus can be achieved. Further, the proposed control schemes do not suffer from the control singularity over the prespecified time interval, thus obviating the use of sinusoid functions[33] or complex switching sliding modes.[24]
- Third, in contrast to those nonsmooth finite/fixed-time consensus laws that use negative power feedback terms[5,7] or the signum function,[18,23,24,29-31] all the proposed FWAT consensus protocols, in the case of no disturbances, are smooth. Further, they are distributed in the sense that information is only communicated locally between neighboring agents; unlike the FWAT average consensus in Reference 28 that uses a deformed Laplacian.
- Fourth, the bound of the convergence time of the proposed consensus schemes is explicitly available, less conservative, and can be chosen arbitrarily regardless of the initial condition or any other parameters, as opposed to those with finite- and fixed-time stability.[14-16,19-21,23] Finally, an application to FWAT formation control of mobile agents is presented.

The remainder of this article is organized as follows. Preliminaries are given in Section 2. Sections 3 and 4 propose FWAT consensus protocols for single- and double-integrator agents, respectively. An application to FWAT formation control of mobile agents is presented in Section 5. Section 6 concludes this article.

2 | PRELIMINARIES

Notation

The set of nonnegative real number is $\mathbb{R}_{\geq 0}$. Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ be the n-dimensional Euclidean space and the $n \times m$ real matrix set, respectively. The vector of all ones is $\mathbf{1}_n$ and the $n \times n$ identity matrix is $\boldsymbol{I}_n$. Given any $\boldsymbol{v} = [v_1, \ldots, v_n]^\top \in \mathbb{R}^n$, denote $|\boldsymbol{v}| = [|v_1|, \ldots, |v_n|]^\top \in \mathbb{R}^n$ and $\mathrm{sgn}(\boldsymbol{v}) = [\mathrm{sgn}(v_1), \ldots, \mathrm{sgn}(v_n)]^\top \in \mathbb{R}^n$, where $\mathrm{sgn}(\cdot)$ denotes the signum function. For any vector $\boldsymbol{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$, we define the diagonal matrix $\mathrm{diag}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ whose the ith diagonal entry is x_i, $\forall i = 1, \ldots, n$, and the other entries are zeros, $\mathrm{e}^{\boldsymbol{x}} = [\mathrm{e}^{x_1}, \ldots, \mathrm{e}^{x_n}]^\top \in \mathbb{R}^n$ and $\ln(\boldsymbol{x}) = [\ln(x_1), \ldots, \ln(x_n)]^\top \in \mathbb{R}^n$. In addition, we also use $\mathrm{e}^{-\mathrm{diag}(\boldsymbol{x})} = \mathrm{diag}(\mathrm{e}^{-\boldsymbol{x}})$ to denote the diagonal matrix with the ith diagonal entry being e^{-x_i}.

2.1 | Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph containing a node set $\mathcal{V} = \{1, \ldots, n\}$, and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ with the cardinality $|\mathcal{E}| = m$. The graph $\mathcal{G}$ is *undirected* if $(i,j) \in \mathcal{E}$ implies that $(j,i) \in \mathcal{E}$; whereas, when $\mathcal{G}$ is *directed*, $(i,j) \in \mathcal{E}$ does not necessarily indicate that $(j,i) \in \mathcal{E}$. If $(i,j) \in \mathcal{E}$ then agent j is a neighbor of agent i. The set of neighbors of agent i is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. The Laplacian matrix $\mathcal{L}(\mathcal{G}) = [l_{ij}] \in \mathbb{R}^{n\times n}$ associated with the graph $\mathcal{G}$ is defined as $l_{ij} = -1$ for $(i,j) \in \mathcal{E},\ i \neq j$, $l_{ii} = -\sum_{j\in\mathcal{N}_i} l_{ij},\ \forall i \in \mathcal{V}$, and $l_{ij} = 0$ otherwise. For an arbitrary orientation of the m edges $\{e_1, \ldots, e_m\}$ in $\mathcal{E}$, we define the incidence matrix $\boldsymbol{H} = [h_{ki}] \in \mathbb{R}^{m\times n}$ as $h_{ki} = 1$ if $e_k = (j,i)$, $h_{ki} = -1$ if $e_k = (i,j)$, and $h_{ki} = 0$ otherwise.

If there is at least one node that is reachable by directed paths from any other nodes, $\mathcal{G}$ is said to contain a (rooted-in) spanning tree. The graph $\mathcal{G}$ is *strongly connected* if there exists a directed path between any two distinct nodes in $\mathcal{V}$. A *leader-following* network consists of a leader and several other follower agents. The leader has no neighbors, whose state thus remains unchanged, and it is reachable from some followers in the system by directed edges. The graph $\mathcal{G}$ is said to be undirected and connected if it contains a spanning tree and each edge in $\mathcal{E}$ is bidirectional. For an undirected and connected graph $\mathcal{G}$, the Laplacian $\mathcal{L}(\mathcal{G})$ is symmetric, positive semidefinite with the eigenvalues being $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_n$. In addition, the eigenvector corresponding to the zero eigenvalue of $\mathcal{L}$ is $\mathbf{1}_n$.[2]

2.2 | Fixed-time stability theory

Consider the following nonlinear dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\alpha}),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ denotes the system state, $\boldsymbol{\alpha} \in \mathbb{R}^l$ contains *adjustable* parameters of (1), and $\boldsymbol{f} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector of nonlinear functions. Let $\boldsymbol{x} = \mathbf{0}$ be an equilibrium point of (1) and $\boldsymbol{x}(t, \boldsymbol{x}_0)$ the solution of (1) starting from an initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$. We now have some definitions.

Definition 1. The origin of (1) is said to be

1. finite-time (FT)[16] stable if it is asymptotic stable and for any $\boldsymbol{x}_0 \in \mathbb{R}^n$ there exists $0 \leq T(\boldsymbol{x}_0, \boldsymbol{\alpha}) < \infty$, called the settling time function, such that $\boldsymbol{x}(t, \boldsymbol{x}_0) = \mathbf{0}$ for all $t \geq t_0 + T(\boldsymbol{x}_0, \boldsymbol{\alpha})$;
2. fixed-time (FxT)[20] stable if it is finite-time stable and there exists $T_{\max}(\boldsymbol{\alpha}) < \infty$ independent of $\boldsymbol{x}_0$ such that $T(\boldsymbol{x}_0, \boldsymbol{\alpha}) \leq T_{\max}(\boldsymbol{\alpha})$;
3. free-will arbitrary time (FWAT)[28] stable if it is fixed-time stable and there exists $0 < T_a < \infty$, which does not depend on $\boldsymbol{x}_0$ or $\boldsymbol{\alpha}$ and can be arbitrarily prespecified, such that $T(\boldsymbol{x}_0, \boldsymbol{\alpha}) \leq T_a$.

The following lemmas are useful to study the prespecified time stability of the origin of (1).

Lemma 1 (27 (Thm. 1)). *Consider the nonlinear system (1) and let $\mathcal{D} \subseteq \mathbb{R}^n$ be a set containing the origin. Let $\beta_1(\boldsymbol{x})$ and $\beta_2(\boldsymbol{x})$ be two continuous positive definite functions on $\mathcal{D}$. Suppose that there exists a real-valued continuously differential function $V(t,\boldsymbol{x}) : [t_0, t_f) \times \mathcal{D} \to \mathbb{R}_{\geq 0}$ and a constant $\eta > 1$ such that*

1. $\beta_1(\boldsymbol{x}) \leq V(t,\boldsymbol{x}) \leq \beta_2(\boldsymbol{x}), \forall t \in [t_0, t_f)$
2. $V(t, \mathbf{0}) = 0, \forall t \in [t_0, t_f)$
3. $\dot{V}(t,\boldsymbol{x}) \leq -\dfrac{\eta}{t_f - t}\left(1 - \mathrm{e}^{-V(t,\boldsymbol{x})}\right), \forall \boldsymbol{x} \in \mathcal{D}, \forall t \in [t_0, t_f)$

then the origin is FWAT stable and $T_a = t_f - t_0$ with t_f being an arbitrary prespecified time. If the equality in (iii) strictly holds for all $\boldsymbol{x} \in \mathcal{D}, \forall t \in [t_0, t_f)$, then the convergence to the origin happens at t_f, that is, $T(\boldsymbol{x}_0, \boldsymbol{\alpha}) = t_f - t_0$.

Lemma 2 (28). *For any $x, y \in \mathbb{R}$ satisfying $0 < x \leq y$, there holds*

$$-x(1 - \mathrm{e}^{-x}) \geq -y(1 - \mathrm{e}^{-y}). \tag{2}$$

Lemma 3 (28 eq.(6)). *For any vector $\boldsymbol{x} \in \mathbb{R}^n$, the following holds*

$$\begin{aligned} -\|\boldsymbol{x}\|\left(1 - \mathrm{e}^{-\|\boldsymbol{x}\|}\right) &\geq -\boldsymbol{x}^\top\left(\boldsymbol{I}_n - \mathrm{e}^{-\mathrm{diag}(\boldsymbol{x})}\right)\mathbf{1}_n \\ &\geq -\boldsymbol{x}^\top\left(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{x}}\right), \end{aligned} \tag{3}$$

where we have used the relation $\mathrm{e}^{-\mathrm{diag}(\boldsymbol{x})}\mathbf{1}_n = \mathrm{e}^{-\boldsymbol{x}}$.

3 | SINGLE-INTEGRATOR AGENTS

This section proposes a distributed FWAT consensus law for a system of multiple agents with single-integrator dynamics, which remedies technical issues associated with the FWAT consensus protocols in Reference 28 (Remark 3). It will be shown that under the proposed control law the multi-agent system achieves an average consensus in an arbitrary prespecified time. When the interaction graph of the system switches between several graph topologies, an arbitrary prespecified time consensus can still be achieved, provided that each of the switching graph topologies is connected (Section 3.3). In the presence of a matched disturbance, a prespecified time consensus scheme based on an integral sliding mode control technique[35] is employed (Section 3.4).

Consider a system of n agents with each agent i maintaining a scalar state x_i. Let $\boldsymbol{x} = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ be the stacked vector of the states of the agents. We adopt the single-integrator model for the dynamics of the agents as follows

$$\dot{\boldsymbol{x}} = \boldsymbol{u}, \ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathbb{R}^n, \tag{4}$$

where $\boldsymbol{u} = [u_1, u_2, \ldots, u_n]^\top \in \mathbb{R}^n$ denotes the control input. We impose the following assumption on the interaction graph of the system.

Assumption 1. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of the system is undirected and connected.

Remark 1. Note that the assumption of undirected graphs $\mathcal{G}$ has often been utilized in the existing works in finite-time, fixed-time, and prespecified time consensus.[12,14,16,20,21,23,26,28,29] Due to the nonlinearity of the consensus protocols and the use of Lyapunov stability analysis, the symmetry, and positive semidefiniteness of the Laplacian matrix $\mathcal{L}(\mathcal{G})$ are crucial for the finite-time convergence analysis in these works. On the other hand, when $\mathcal{G}$ is directed, $\mathcal{L}(\mathcal{G})$ becomes non-symmetric, that is, $\mathcal{L} \neq \mathcal{L}^\top$, and it is not positive semidefinite. Therefore, it is not straightforward to analyze the stability of the nonlinear consensus systems. Few works have addressed fixed-time[21,33] and prespecified time[12] consensus for MASs with either leader-following or strongly connected graphs.

3.1 | Proposed average consensus laws

We propose the following FWAT average consensus for the system

$$\boldsymbol{u}=\begin{cases}\frac{\eta}{t_f-t}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}}, & \text{if } t_0 \le t < t_f\\ \boldsymbol{0}, & \text{otherwise,}\end{cases} \tag{5}$$

for a positive constant $\eta > 1/\lambda_2^2(\mathcal{L})$, where $\mathcal{L} \in \mathbb{R}^{n\times n}$ is the Laplacian matrix of the graph $\mathcal{G}$, and $t_f > t_0$ is an arbitrary prespecified time. The control law for each agent i in (5) is explicitly given as

$$u_i = \frac{\eta}{t_f - t}\sum_{j\in\mathcal{N}_i}\left\{\exp\left[\sum_{j\in\mathcal{N}_i}(x_j - x_i)\right] - \exp\left[\sum_{k\in\mathcal{N}_j}(x_k - x_j)\right]\right\}.$$

Therefore, each agent i needs to communicate a sum of the relative states $z_i := \sum_{j\in\mathcal{N}_i}(x_j - x_i)$ to its neighbors. In many coordination control scenarios related to MASs, see for example, References 36 and 37, the agents sense relative states, such as relative positions[38] or relative bearing vectors,[5] to their neighbors; thus, in these cases, such a z_i is readily available to each agent i for communication, assuming no communication delays. Each agent i then simply broadcasts z_i to its neighbors $j \in \mathcal{N}_i$ in order to carry out (5). The FWAT consensus scheme is summarized in Algorithm 1.

Algorithm 1. Free-will arbitrary time consensus scheme

Initialization: $t \leftarrow t_0,\ \boldsymbol{x}(t_0) \leftarrow \boldsymbol{x}_0 \in \mathbb{R}^n,\ \eta{>}1/\lambda_2^2,\ t_f{>}t_0$.
Consensus control loop:
  while $t{<}t_f$ do
    for each agent $i \in \mathcal{V}$ **do**
      Sense relative states $(x_j - x_i)$ to neighbors $j \in \mathcal{N}_i$ and compute $z_i = \sum_{j\in\mathcal{N}_i}(x_j - x_i)$.
      Communicate z_i to, and receive z_j from, neighbors $j \in \mathcal{N}_i$.
      $\boldsymbol{x}_i(t) \leftarrow$ integrate $\dot{x}_i(t) = \frac{\eta}{t_f-t}\sum_{j\in\mathcal{N}_i}\left(\exp(z_i) - \exp(z_j)\right)$.
    end for
  end while.
End consensus control loop.

Remark 2. To avoid possible communication of the states of two-hop neighbors of the agents, the following FWAT consensus law for first-order MASs, proposed in the work,[29] can be employed:

$$\boldsymbol{u}(t)=\begin{cases}-\frac{\eta_1}{t_f-t}\boldsymbol{H}^\top \mathrm{diag}\left(\mathrm{sgn}(\boldsymbol{H}\boldsymbol{x})\right)\left(\boldsymbol{1}_m - \mathrm{e}^{-|\boldsymbol{H}\boldsymbol{x}|}\right), & \text{if } t_0 \le t < t_f\\ \boldsymbol{0}, & \text{otherwise,}\end{cases} \tag{6}$$

where η_1 is a sufficiently large constant, $\boldsymbol{H} \in \mathbb{R}^{m\times n}$ denotes the incidence matrix of $\mathcal{G}$, and $m = |\mathcal{E}|$ is the number of edges in $\mathcal{E}$. Indeed, the control for each agent i is given as

$$u_i = \frac{\eta_1}{t_f - t}\sum_{j\in\mathcal{N}_i}\mathrm{diag}\left(\mathrm{sgn}(x_j - x_i)\right)\left(1 - \exp(-|x_j - x_i|)\right),$$

which requires communication of only one-hop neighbors' states. Results analogous to the FWAT average consensus of the system with first- and second-order dynamics under the nominal consensus protocol (5) in the following sections can also be obtained similarly for (6).

Remark 3. In Reference 28, a consensus protocol was proposed as

$$\boldsymbol{u} = \begin{cases} -\frac{\eta}{t_f - t}\left(\boldsymbol{I}_n - \mathrm{e}^{-\mathrm{diag}(\mathcal{L}\boldsymbol{x})}\right)\mathbf{1}_n, & \text{if } t_0 \le t < t_f \\ \mathbf{0}, & \text{otherwise,} \end{cases} \tag{7}$$

where η is a positive constant such that $\eta > 1/\lambda_2(\mathcal{L})$. It is stated in Reference 28 (Theorem 2) that under Assumption 1 and consensus law (7), the agents achieve a consensus at an arbitrary chosen time t_f. The proof of Reference 28 (Theorem 2) relies on the following inequality

$$\lambda_2||\boldsymbol{x}||^2 \le \boldsymbol{x}^\top \mathcal{L}\boldsymbol{x}. \tag{8}$$

This inequality is however not correct since the Laplacian matrix $\mathcal{L}$ is only positive semidefinite. Indeed, by selecting $\boldsymbol{x} = \mathbf{1}_n$ and using the relation $\mathcal{L}\mathbf{1}_n = \mathbf{0}$, one has

$$\lambda_2||\mathbf{1}_n||^2 = \lambda_2 n > 0 = \mathbf{1}_n^\top \mathcal{L}\mathbf{1}_n,$$

which is a contradiction. If for any vector $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{y} \perp \mathrm{null}(\mathcal{L}) = \mathrm{span}(\mathbf{1}_n)$ we can only have a corresponding relation $\lambda_2||\boldsymbol{y}||^2 \le \boldsymbol{y}^\top \mathcal{L}\boldsymbol{y}$.

To achieve an average consensus,[28] proposed an alternative consensus law in Reference 28 (eq. 24). It is noted that the deformed Laplacian used in the average consensus law[28eq.(24)] is not for diffusive coupling. Moreover, the consensus law requires that all agents know the average of their initial states $\bar{x} := \sum_{i=1}^n x_i(t_0)/n$. This requirement is restrictive since the average $\bar{x}$ is not readily available to the agents, and the initial state vector $\boldsymbol{x}(t_0)$ might be initialized arbitrarily. The distributed nature of the average consensus scheme[28(eq.24)] is therefore questionable.

It can be verified that our proposed consensus law (5) is modified from (7) by left-multiplying by $\mathcal{L}$ on the right hand side of (7). Indeed, we have that

$$-\frac{\eta}{t_f - t}\mathcal{L}\left(\boldsymbol{I}_n - \mathrm{e}^{-\mathrm{diag}(\mathcal{L}\boldsymbol{x})}\right)\mathbf{1}_n = -\frac{\eta}{t_f - t}\left(\mathcal{L}\mathbf{1}_n - \mathcal{L}\mathrm{e}^{-\mathrm{diag}(\mathcal{L}\boldsymbol{x})}\mathbf{1}_n\right) = \frac{\eta}{t_f - t}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}},$$

which is identical to (5), where in the last equality we have used the relations $\mathcal{L}\mathbf{1}_n = \mathbf{0}$ and $\mathrm{e}^{-\mathrm{diag}(\mathcal{L}\boldsymbol{x})}\mathbf{1}_n = \mathrm{e}^{-\mathcal{L}\boldsymbol{x}}$. More importantly, the proposed (*distributed*) average consensus law (5) overcomes all the aforementioned technical issues associated with (7).

A free-will arbitrary time convergence of the system under (5) is given in the following subsection.

3.2 | Convergence analysis

Denote $\bar{x} := \mathbf{1}_n^\top \boldsymbol{x}(t_0)/n$ as the average of the agents' initial states. Let $\delta_i(t) := (x_i(t) - \bar{x}) \in \mathbb{R}$ and define $\boldsymbol{\delta}(t) := [\delta_1, \dots, \delta_n]^\top = \boldsymbol{x}(t) - \bar{x}\mathbf{1}_n \in \mathbb{R}^n$ as the error vector. It then follows that $\dot{\boldsymbol{x}}(t) = \dot{\boldsymbol{\delta}}(t)$, and $\mathcal{L}\boldsymbol{\delta} = \mathcal{L}(\boldsymbol{x} - \bar{x}\mathbf{1}_n) = \mathcal{L}\boldsymbol{x}$ since $\mathcal{L}\mathbf{1}_n = \mathbf{0}$ due to Assumption 1. In the sequel, we study the convergence of the proposed consensus protocol (5).

We first show that the average of the agent states $\frac{\mathbf{1}_n^\top \boldsymbol{x}(t)}{n}$ along the trajectory of (5) is time-invariant.

Lemma 4. *Assume that Assumption 1 holds. Under the consensus law (5), the average of the agent states $\mathbf{1}_n^\top \boldsymbol{x}(t)/n$ is time-invariant.*

Proof. Since $\mathbf{1}_n^\top \mathcal{L} = \mathbf{0}$ we have that $\frac{d}{dt}\left(\sum_{i=1}^n x_i\right) = \mathbf{1}_n^\top \dot{\boldsymbol{x}} = \frac{\eta}{t_f - t}\mathbf{1}_n^\top \mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}} = 0$ along the trajectory of (5). It follows that the average of the agent states, that is, $\mathbf{1}_n^\top \boldsymbol{x}(t)/n = \sum_{i=1}^n x_i(t)/n$, is time-invariant. ■

Moreover, the dynamics of the error vector $\boldsymbol{\delta}(t) = \boldsymbol{x}(t) - \bar{x}\mathbf{1}_n$, where $\bar{x}$ is the average of the agents' initial states, is given as

$$\dot{\boldsymbol{\delta}} = \begin{cases} -\frac{\eta}{t_f - t}\mathcal{L}\left(\mathbf{1}_n - \mathrm{e}^{-\mathcal{L}\boldsymbol{\delta}}\right), & \text{if } t_0 \le t < t_f \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

We can now prove the following result.

Theorem 1. *Assume that Assumption 1 holds. Under the consensus law (5) with $\eta > 1/\lambda_2^2$, $\boldsymbol{x}(t)$ converges to $\mathbf{1}_n\bar{x}$ within the chosen settling time $T_a = t_f - t_0$.*

Proof. Consider the Lyapunov function

$$V = \boldsymbol{\delta}^\top \boldsymbol{\delta}, \tag{10}$$

which is positive definite, radially unbounded and continuously differentiable in $t_0 \leq t < t_f$. The derivative of V along a trajectory of (9) is given as

$$\begin{aligned} \dot{V} &= 2\boldsymbol{\delta}^\top \dot{\boldsymbol{\delta}} \\ &= -\frac{2\eta}{t_f - t}\boldsymbol{\delta}^\top \mathcal{L}\left(\mathbf{1}_n - \mathrm{e}^{-\mathcal{L}\boldsymbol{\delta}}\right) \\ &= -\frac{2\eta}{t_f - t}(\mathcal{L}\boldsymbol{\delta})^\top\left(\mathbf{1}_n - \mathrm{e}^{-\mathcal{L}\boldsymbol{\delta}}\right) \\ &\leq -\frac{2\eta}{t_f - t}\|\mathcal{L}\boldsymbol{\delta}\|\left(1 - \mathrm{e}^{-\|\mathcal{L}\boldsymbol{\delta}\|}\right), \end{aligned} \tag{11}$$

where the third equality follows from the symmetry of Laplacian matrix $\mathcal{L}^\top = \mathcal{L}$ due to the undirected nature of the graph $\mathcal{G}$, and in the last inequality we have used Lemma 3.

Since $\mathbf{1}_n^\top \boldsymbol{x}(t)$ is time-invariant (Lemma 4) one has $\mathbf{1}_n^\top \boldsymbol{\delta} = \mathbf{1}_n^\top(\boldsymbol{x} - \bar{x}\mathbf{1}_n) = 0$ for all $t \geq t_0$. In other words, $\boldsymbol{\delta}$ is orthogonal to the eigenvector $\mathbf{1}_n$ corresponding to the zero eigenvalue of $\mathcal{L}$ for all $t \geq t_0$. Consequently, we have that

$$\begin{aligned} \lambda_2(\mathcal{L})||\boldsymbol{\delta}||^2 &\leq \delta^\top \mathcal{L}\boldsymbol{\delta} \\ &\leq \|\boldsymbol{\delta}\|\|\mathcal{L}\boldsymbol{\delta}\| \\ \Leftrightarrow \lambda_2(\mathcal{L})\sqrt{V} &\leq \|\mathcal{L}\boldsymbol{\delta}\|, \end{aligned} \tag{12}$$

where the second inequality follows from Holder's inequality. From the preceding inequality, Lemma 2 and (11), we obtain that

$$\dot{V} \leq -\frac{2\eta\lambda_2}{t_f - t}\sqrt{V}\left(1 - \mathrm{e}^{-\lambda_2\sqrt{V}}\right). \tag{13}$$

Let $\xi := \lambda_2\sqrt{V} \geq 0$. Then, one obtains

$$\dot{\xi}(t) = \lambda_2 \frac{\dot{V}}{2\sqrt{V}} \overset{(13)}{\leq} -\frac{\eta\lambda_2^2}{t_f - t}\left(1 - \mathrm{e}^{-\xi(t)}\right). \tag{14}$$

Let $\bar{\eta} := \eta\lambda_2^2 > 1$ and consider the following first-order differential equation

$$\dot{\zeta}(t) = -\frac{\bar{\eta}}{t_f - t}\left(1 - \mathrm{e}^{-\zeta(t)}\right),\ t \in [t_0, t_f),\ (\text{and } \dot{\zeta}(t) = 0 \text{ if } t \geq t_f), \tag{15}$$

with $\zeta(0) = \xi(0) \geq 0$. The solution to the preceding system is

$$\zeta(t) = \ln\left(1 + \alpha(t_f - t)^{\bar{\eta}}\right),\ t \in [t_0, t_f),$$

where $\alpha = \frac{(\mathrm{e}^{\zeta(t_0)} - 1)}{(t_f - t_0)^{\bar{\eta}}}$ is a constant. Therefore, one has $\zeta(t_f) = 0$ and

$$\dot{\zeta}(t) = \frac{-\bar{\eta}\alpha(t_f - t)^{\bar{\eta}-1}}{1 + \alpha(t_f - t)^{\bar{\eta}}}. \tag{16}$$

It follows from (15) and (16) that $\dot{\zeta}(t_f) = 0$ (since $\dot{\zeta}(t_f^+) = \dot{\zeta}(t_f^-) = 0$). As a result, $\zeta(t) = 0$ is maintained for all $t \geq t_f$. Furthermore, by (14) and the comparison lemma[39(Lm. 3.4)], one obtains $\xi(t) \leq \zeta(t)$ for all $t \in [t_0, t_f)$. Consequently, $\xi(t)$ converges to zero within a free-will arbitrary settling time $T_a = (t_f - t_0)$ and so does $V(t)$. As a result, $\boldsymbol{\delta} = \mathbf{0}$ or $x_i = \overline{x} = \sum_{i=1}^{n} x_i(0)/n, \forall i \in \mathcal{V}$, for all $t \geq t_f$. ■

Note that though the design parameter η needs to be not too small to ensure a prespecified time consensus under (5), the prespecified settling time $T_a = t_f - t_0$ is independent of the initial states or any other design parameters. Further, the parameter η does not affect the settling time, and hence can be chosen just slightly above $1/\lambda_2^2$ to ensure the stability of the system. It is also obvious from (5) that a smaller settling time $(t_f - t_0)$ leads to higher control inputs.

Remark 4. The proof of Theorem 1 relies on the undirectedness of the graph $\mathcal{G}$, for which the Laplacian $\mathcal{L}(\mathcal{G})$ is symmetric, that is, $\mathcal{L}^\top = \mathcal{L}$, and positive definite in the orthogonal complement of span$\{\mathbf{1}_n\}$, that is, $\delta^\top \mathcal{L}\boldsymbol{\delta} \geq \lambda_2(\mathcal{L})||\boldsymbol{\delta}||^2, \forall \boldsymbol{\delta} \perp \mathbf{1}_n$.

Consider a leader-following network consisting of one leader and n followers, with the graph of the followers, denoted as $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f), \mathcal{V}_f := \{1, \ldots, n\}$, being undirected. Let $\boldsymbol{B} = \mathrm{diag}(\{b_i\}_{i \in \mathcal{V}_f})$, where $b_i = 1$ if there is a directed edge from follower i to the leader, and $b_i = 0$ otherwise. It then can be shown that $(B + \mathcal{L}_f(\mathcal{G}_f))$ is a *symmetric positive definite* matrix,[34] where $\mathcal{L}_f$ is the Laplacian of $\mathcal{G}_f$. Thus, under the following consensus law, for each $i \in \mathcal{V}_f$,

$$u_i = \frac{\eta}{t_f - t}\left\{1 - \exp\left[\sum_{j \in \mathcal{N}_i}(x_j - x_i)\right]\right\},\ t \in [t_0, t_f),\ \eta > 1,\ \text{and } u_i = 0 \text{ otherwise},$$

the followers' states converge to the leader's state, denoted as x_0, in free-will arbitrary time. Indeed, let $\delta_i(t) := (x_i(t) - x_0) \in \mathbb{R}$ and define $\boldsymbol{\delta}(t) := [\delta_1, \ldots, \delta_n]^\top$. Then the error dynamics is given as[*]

$$\dot{\boldsymbol{\delta}} = -\frac{\eta}{t_f - t}\left(\boldsymbol{I}_n - \mathrm{e}^{-\mathrm{diag}((\boldsymbol{B}+\mathcal{L}_f)\boldsymbol{\delta})}\right)\mathbf{1}_n, t \in [t_0, t_f).$$

Following similar lines as in Proof of Theorem 1, one can show that $\boldsymbol{\delta} \to \mathbf{0}$ as $t \to t_f$. If the leader-following graph is acyclic (having no loop) and contains a spanning tree (rooted at the leader node), we assume that each follower i selects a preset time in the preceding controller, namely $t_{fi} > 0$, such that it is larger than that of its neighbors, that is, $t_{fi} > \max\{t_{fj}\}_{j \in \mathcal{N}_i}$. Therefore, it can be proved that the followers' states converge to x_0 within the prespecified times in a sequential order along the spanning tree.

However, if the graph $\mathcal{G}$ is a general directed graph, it is not straightforward to show the FWAT convergence of the system.

3.3 | FWAT consensus under switching graph topologies

This subsection considers FWAT consensus of MASs under switching graphs. Let us assume that the graph of the system is time-varying and is denoted by $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ with $\mathcal{E}_{\sigma(t)} \subseteq \mathcal{V} \times \mathcal{V}$ and $\sigma(t) : \mathbb{R}_{\geq 0} \to \mathcal{P} := \{1, 2, \ldots, \rho\}$ being a piecewise constant switching signal, where $\rho = |\mathcal{P}|$ denotes the number of switching graphs. It is assumed that there exists a sequence of time instants $\{t_k\}, k \in \mathbb{Z}^+$ such that $\sigma(t)$ is a constant for each successive time interval $t \in [t_k, t_{k+1})$, $t_{k+1} - t_k > \tau_s > 0, \forall k$. We assume the following uniform connectedness condition.

Assumption 2. Each graph topology $\mathcal{G}_k, \forall k \in \mathcal{P}$ is undirected and connected.

As a result, the Laplacian $\mathcal{L}_{\sigma(t)}$ associated with the graph $\mathcal{G}_{\sigma(t)}$ remains positive semidefinite with $\lambda_2(\mathcal{L}_{\sigma(t)})$ being strictly positive, for all $t \geq t_0$. Let $\overline{\lambda}_2 := \min\{\lambda_2(\mathcal{L}_\sigma)\}_{\sigma \in \mathcal{P}}$ be the smallest nonzero eigenvalue of the Laplacian matrix among the switch graph topologies. Then, for any error vector $\boldsymbol{\delta} \in \mathbb{R}^n$ satisfying $\delta \perp \mathbf{1}_n$, the following inequality still holds

$$\overline{\lambda}_2||\boldsymbol{\delta}||^2 \leq \boldsymbol{\delta}^\top \mathcal{L}_{\sigma(t)}\boldsymbol{\delta},\ \forall t \geq t_0. \tag{17}$$

Thus, we obtain the following result whose proof can be shown by following similar lines as in Proof of Theorem 1.

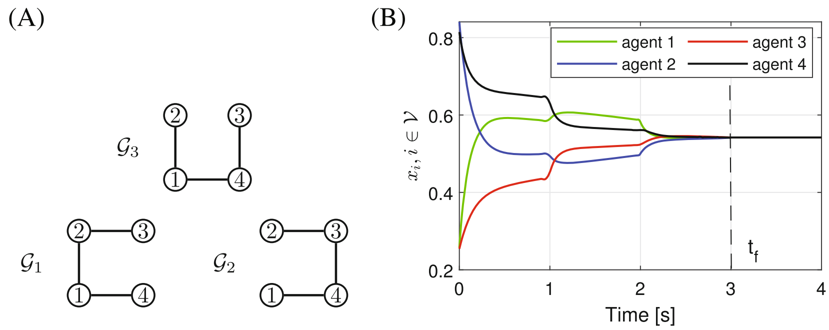

FIGURE 1 FWAT consensus of four agents under switching graphs $\mathcal{G}_\sigma$, $\sigma = 1, 2, 3$. Each graph topology is undirected and connected. (A) Switching graphs $\mathcal{G}_\sigma$. (B) Evolutions of the agents' states versus time

Corollary 1. *Consider the multi-agent system (4) with switching graph topologies $\mathcal{G}_{\sigma(t)}$ satisfying Assumption 2. Under the consensus law (5) with $\eta > 1/\overline{\lambda}_2^2$, $\boldsymbol{x}(t)$ converges to $\mathbf{1}_n\bar{x}$, or equivalently each x_i converges to the initial state average $\bar{x}$, in fixed time with the prespecified settling time $T_a = t_f - t_0$.*

Since the consensus law is fixed-time convergent and t_f is independent of the initial state (or any other design parameters), we may allow the graph to be empty for some time interval $[t_1, t_2] \subset [t_0, t_f)$. That is, sometimes, all nodes may be disconnected from the network for a short amount of time and then reconnected. The free-will arbitrary time convergence property allows the consensus to be still achieved at some time instant $t \leq t_f$.

Example 1:

Consider a system of four agents whose communication graph switches every 0.5 s between the three graph topologies $\{\mathcal{G}_\sigma\}_{\sigma=1,2,3}$ given in Figure 1A. The agents' initial states are chosen in $[0, 1]$. Simulation results for FWAT consensus of the agents under the FWAT consensus law (5) with $t_f = 3$ s and $\eta = 3$ are given in Figure 1B. We observe that the agents achieve the average consensus within the chosen time t_f.

Example 2:

Consider a system of four agents with the interaction graph given as $\mathcal{G}_1$ in Figure 1A. The initial state vector is $\boldsymbol{x}(0) = [0.258, 0.84, 0.254, 0.814]^\top$. Simulation results are provided in Figure 2 for the consensus of the system under the control law (5) with $t_f = 2$ and 0.08 s, and $\eta = 3$ ($\eta > 1/\lambda_2^2(\mathcal{G}_1) = 2.9141$), and the following finite-time (FT) consensus law[14]

$$\boldsymbol{u} = -\boldsymbol{H}^\top \mathrm{diag}(\mathrm{sgn}(\boldsymbol{Hx}))|\boldsymbol{Hx}|^\alpha, \ \alpha \in (0, 1). \tag{18}$$

The control parameter was chosen as $\alpha = 0.5$. As can be seen from Figure 2A,B, the settling time t_f of the average consensus under the proposed FWAT consensus law can be arbitrarily selected. On the contrary, the settling time of the system under (18) is not easily adjustable since it depends on the control and network parameters, and the initial states.[14] Moreover, such estimates of the bound of the convergence time (i.e., the settling time) in finite- or even fixed-time consensus are very conservative, as observed in References 22 and 23. As shown in Figure 2C, the states converge to the average consensus after about 1.3 s, while, the theoretical bound of the settling time is computed as $T_{FT} \approx 2.26$ s. Furthermore, though the control inputs of the FT consensus law (18) are smaller than that of (5), they continue to fluctuate wildly after the convergence time due to the use of the signum function.

We also observe from the simulations that selecting a smaller value of t_f leads to higher control inputs of the agents in the transient state (see Figure 2D,E). Thus, one needs to select t_f not too small to avoid large control inputs. In practice, the control input of an agent can be physically bounded, so if a smaller settling time is selected then its actuator cannot provide the desired action which is very large, hence the convergence of the system might be affected, which deserves further investigation.

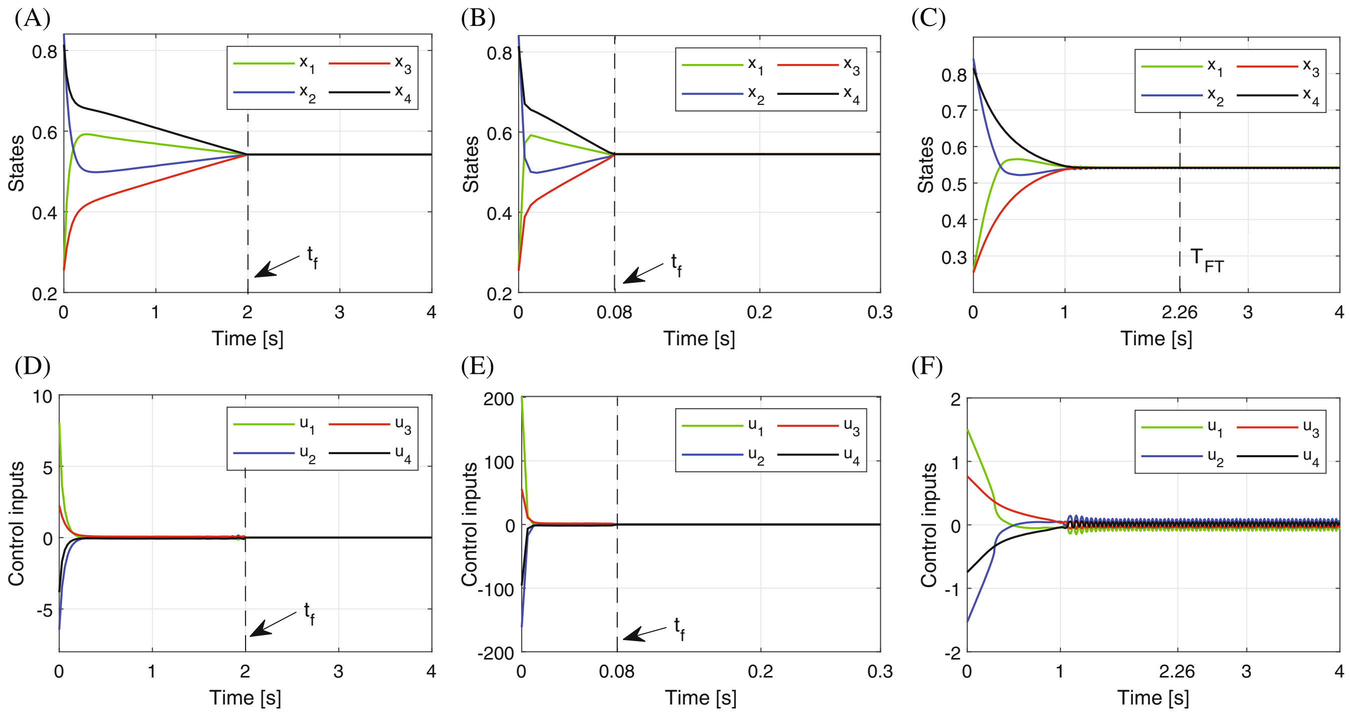

FIGURE 2 Consensus of four agents respectively under the FWAT control law (5) with $t_f = 2$ s (A) and 0.08 s (B), and the FT consensus law (18) with $\alpha = 0.5$ (C). The corresponding control inputs versus time (D–F)

3.4 | Free-will arbitrary time consensus under disturbances

Consider the case that the dynamics (4) of the agents is affected by disturbance as follows

$$\dot{x}_i = u_i + d_i,\ x_i(t_0) \in \mathbb{R},\ \ i = 1, \ldots, n, \tag{19}$$

where $d_i(t)$ is a (time-varying) matched disturbance associated with agent i. The disturbances might result from the model uncertainties and external disturbances.[18,23,34] We assume that the disturbances of the agents are all bounded, which can also represent stochastic disturbances.[40] Thus, there exists a positive constant $\overline{d}$ such that $\max |d_i| \leq \overline{d}$ for all $i \in \mathcal{V}$. Following similar techniques that are used to design integral sliding mode controllers as in References 18,28, and 35, we can design the sliding surface $\boldsymbol{s} \in \mathbb{R}^n$ as

$$\boldsymbol{s} = \boldsymbol{x} - \boldsymbol{x}_0 - \int_{t_0}^{t} \boldsymbol{u}^{nom}(\tau)d\tau, \tag{20}$$

where the nominal control $\boldsymbol{u}^{nom}$ is given as before in (5) and $\boldsymbol{x}_0 = [x_1(t_0), \ldots, x_n(t_0)]^\top$. Note importantly that $\boldsymbol{s}(t_0) = \boldsymbol{0}$, or equivalently the system trajectory lies on the sliding surface at t_0. Now if the consensus law of the system is designed as

$$\boldsymbol{u} = \boldsymbol{u}^{nom} - \beta \operatorname{sgn}(\boldsymbol{s}), \tag{21}$$

(or alternatively, $\boldsymbol{u} = \boldsymbol{u}^{nom} - \beta \frac{\boldsymbol{s}}{\|\boldsymbol{s}\|}$) with the control gain $\beta > \overline{d}$. Due to the discontinuous sgn term (resp. the negative power feedback term $\|\boldsymbol{s}\|^{-1}$) in the control law, the solution trajectory will be understood in the sense of Filippov.[41] It is then not hard to see that $\boldsymbol{s} \equiv \boldsymbol{0}$ for all $t \geq t_0$,[18,28,30,35] and hence the trajectory is kept on the sliding surface at t_0 and thereafter. On the sliding surface, we have $\dot{\boldsymbol{s}} = \boldsymbol{0} \Rightarrow \dot{\boldsymbol{x}} = \boldsymbol{u}^{nom}$, which follows the nominal FWAT time stable system (4) and (5), and thus a prespecified time average consensus is achieved by the agents (Theorem 1).

4 | DOUBLE-INTEGRATOR AGENTS

This section proposes a FWAT consensus protocol for systems of double-integrator modeled agents. To this end, a FWAT tracking control scheme is presented to reduce the second-order system to the first-order counterpart, which has been shown to achieve a consensus in an arbitrary prespecified time in the previous section. The agents' states are also shown to be bounded during the transient time of the tracking error system. A consensus scheme for a FWAT consensus of p-order integrator agents ($p > 2$) is also investigated.

Consider the system of n agents whose dynamics is modeled as the second-order system

$$\dot{\boldsymbol{x}} = \boldsymbol{v},\ \ \dot{\boldsymbol{v}} = \boldsymbol{u}, \tag{22}$$

where $\boldsymbol{v} = [v_1, \dots, v_n]^\top \in \mathbb{R}^n$ denotes the velocity vector and $\boldsymbol{u} \in \mathbb{R}^n$ is the control vector. We consider the change of variable

$$\boldsymbol{z} = (\boldsymbol{v} + \boldsymbol{\phi}_1) \in \mathbb{R}^n, \tag{23}$$

where

$$-\boldsymbol{\phi}_1(\boldsymbol{z}, t) := \frac{\eta}{t_f - t}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}},$$

is the time-varying desired vector that we want the velocity vector $\boldsymbol{v} \in \mathbb{R}^n$ to track. Note that the newly introduced auxiliary vector $\boldsymbol{z} \in \mathbb{R}^n$ is only used for simplicity of representation and analysis, as shown in the below. A possible approach is first steering the velocity $\boldsymbol{v}(t)$ to track the vector $-\boldsymbol{\phi}_1(t)$ in a free will arbitrary prespecified time $t_1 > 0$ (satisfying $t_1 < t_f$), and then treating (22) as the reduced single-integrator model $\dot{\boldsymbol{x}} = -\boldsymbol{\phi}_1$ thereafter, provided that the system state is bounded in $t \in [t_0, t_1]$.

4.1 | Proposed consensus law

To proceed, the time derivative of the vector $\boldsymbol{z}$ in (23) is given as

$$\dot{\boldsymbol{z}} = \dot{\boldsymbol{v}} + \frac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{x}}\boldsymbol{v} + \frac{\partial \boldsymbol{\phi}_1}{\partial t}, \tag{24}$$

$$= \dot{\boldsymbol{v}} + \frac{\eta}{t_f - t}\mathcal{L}\mathrm{diag}\left(\mathrm{e}^{-\mathcal{L}\boldsymbol{x}}\right)\mathcal{L}\boldsymbol{v} - \frac{\eta}{(t_f - t)^2}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}}. \tag{25}$$

By canceling out the first-order derivative terms on the right hand side of (25) above, we design the control input as

$$\boldsymbol{u} = \begin{cases} -\dfrac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{x}}\boldsymbol{v} - \dfrac{\partial \boldsymbol{\phi}_1}{\partial t} - \dfrac{\eta_2}{t_1 - t}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}), & \text{if } t_0 \le t < t_1 \\ -\dfrac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{x}}\boldsymbol{v} - \dfrac{\partial \boldsymbol{\phi}_1}{\partial t}, & \text{if } t_1 \le t < t_f \\ \mathbf{0}, & \text{otherwise,} \end{cases} \tag{26}$$

where $0 < t_1 < t_f$ and $\eta_2 > 1$. From (25) and (26), each agent i needs to communicate the sum of the relative states $\sum_{j \in \mathcal{N}_i}(x_j - x_i)$ and the sum of the relative velocities $\sum_{j \in \mathcal{N}_i}(v_j - v_i)$ to its neighbors. Thus, the proposed consensus law (26) for second order system (22) is also distributed. In addition, to avoid high control inputs in the transient state, the values of $t_1 - t_0$ and $t_f - t_1$ should not be chosen too small.

We can now state the main result of this section.

Theorem 2. *Consider the system of double-integrator agents (22) with a connected graph $\mathcal{G}$. Under the consensus law (26) with $\eta > 1/\lambda_2^2$ and $\eta_2 > 1$, $\boldsymbol{v}(t) \to \mathbf{0}$ and $\boldsymbol{x}(t)$ converges to a consensus in fixed time with the settling time $T_a = t_f - t_0$.*

Proof. Let us consider the Lyapunov function

$$V_2(\boldsymbol{z}) = \boldsymbol{z}^\top \boldsymbol{z}. \tag{27}$$

The derivative of V_2 along the trajectory of (26) is given as

$$\begin{aligned}
\dot{V}_2 &= 2\boldsymbol{z}^\top \dot{\boldsymbol{z}} \\
&= -2\frac{\eta_2}{t_1 - t}\boldsymbol{z}^\top(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) \\
&\overset{(3)}{\leq} -2\frac{\eta_2}{t_1 - t}||\boldsymbol{z}||\left(1 - \mathrm{e}^{-||\boldsymbol{z}||}\right) \\
&\leq -2\frac{\eta_2}{t_1 - t}\sqrt{V_2}\left(1 - \mathrm{e}^{-\sqrt{V_2}}\right).
\end{aligned}$$

Let $\xi = \sqrt{V_2} = ||\boldsymbol{z}||$. Then, one has

$$\dot{\xi} = \frac{\dot{V}_2}{2\sqrt{V_2}} \leq -\frac{\eta}{t_1 - t}(1 - \mathrm{e}^{-\xi}), \tag{28}$$

which is in the form of the free-will arbitrary time convergent system in (14). As a result, $\boldsymbol{z} \to \mathbf{0}$ asymptotically as $t \to t_1$ or equivalently $\boldsymbol{v}(t) = -\boldsymbol{\phi}_1$ for all time $t \geq t_1$. Further, the state vector $\boldsymbol{x}(t)$ is bounded for all time $t \in [t_0, t_f]$ (see Lemma 7 in Section 4.2 below). Therefore, the system (22) is reduced to the following single-integrator dynamics

$$\dot{\boldsymbol{x}} = \frac{\eta}{t_f - t}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}}, \ \forall t \geq t_1. \tag{29}$$

As a result, the agents's states converges to the average of the agents' states at $t = t_1$, namely $\bar{x}(t_1) := \mathbf{1}_n^\top \boldsymbol{x}(t_1)/n$, within the prespecified time t_f, if $\eta > 1/\lambda_2^2$ (Theorem 1). Since $\boldsymbol{v}(t) = -\boldsymbol{\phi}_1$ for all $t \geq t_1$, and $\boldsymbol{\phi}_1 \to \mathbf{0}$ as $\boldsymbol{x} \to \bar{x}(t_1)$, we conclude that the velocity vector of the system $\boldsymbol{v} \to \mathbf{0}$ at the same time instant as $\boldsymbol{x} \to \mathbf{1}_n\bar{x}(t_1)$. ■

For the sake of completeness, we clarify below that (26) is smooth at $t = t_1$, and investigate the system behavior during the time interval $[t_0, t_1]$ in the following subsection.

Lemma 5. *For any $\eta_2 > 1$, the FWAT consensus law (26) is smooth at $t = t_1$.*

Proof. By (24) and (26) we have $\dot{\boldsymbol{z}} = -\frac{\eta_2}{t_1 - t}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}})$ if $t \in [t_0, t_1)$, and $\dot{\boldsymbol{z}} \equiv \mathbf{0}$ if $t \geq t_1$, and hence one can obtain

$$\boldsymbol{z}(t) = \ln\left(\mathbf{1}_n + \boldsymbol{c}(t_1 - t)^{\eta_2}\right), \ t \in [t_0, t_1),$$

where $\boldsymbol{c} = [c_1, \ldots, c_n]^\top := \frac{(\mathrm{e}^{\boldsymbol{z}(t_0)} - \mathbf{1}_n)}{(t_1 - t_0)^{\eta_2}} \in \mathbb{R}^n$. Let $\boldsymbol{\psi}(\boldsymbol{z}, t) := -\frac{\eta_2}{t_1 - t}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}})$ if $t \in [t_0, t_1)$, and $\boldsymbol{\psi}(\boldsymbol{z}, t) \equiv \mathbf{0}$ if $t \geq t_1$. Therefore, we have that

$$\begin{aligned}
\dot{\boldsymbol{\psi}}(t) &= \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{z}}\dot{\boldsymbol{z}} + \frac{\partial \boldsymbol{\psi}}{\partial t} \\
&= \frac{\eta_2^2}{(t_1 - t)^2}\mathrm{diag}\left(\mathrm{e}^{-\boldsymbol{z}}\right)(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) - \frac{\eta_2^2}{(t_1 - t)^2}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) \\
&= -\frac{\eta_2^2}{(t_1 - t)^2}\mathrm{diag}\left(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}\right)(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) \\
&= -\eta_2^2\left[\frac{c_1^2(t_1 - t)^{2(\eta_2 - 1)}}{(1 + c_1(t_1 - t)^{\eta_2})^2}, \ldots, \frac{c_n^2(t_1 - t)^{2(\eta_2 - 1)}}{(1 + c_n(t_1 - t)^{\eta_2})^2}\right]^\top.
\end{aligned}$$

It follows that for any $\eta_2 > 1$, $\dot{\boldsymbol{\psi}}(t = t_1^-) = \mathbf{0}$. It can also be computed that

$$\frac{d^p}{dt^p}\boldsymbol{\psi}(t) = -\frac{p!\,\eta_2^{p+1}}{(t_1-t)^{p+1}}\mathrm{diag}^p\left(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}\right)(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}})$$
$$= -p!\,\eta_2^{p+1}\left[\frac{c_1^{p+1}(t_1-t)^{(p+1)(\eta_2-1)}}{(1+c_1(t_1-t)^{\eta_2})^{p+1}}, \ \dots, \ \frac{c_n^{p+1}(t_1-t)^{(p+1)(\eta_2-1)}}{(1+c_n(t_1-t)^{\eta_2})^{p+1}}\right]^\top.$$

Therefore, $\frac{d^p}{dt^p}\boldsymbol{\psi}(t)\Big|_{t=t_1^-} = \mathbf{0}$ for all $p = 1, 2, \dots,$ if $\eta_2 > 1$. Consequently, (26) is smooth at $t = t_1$ since $\lim_{t\to t_1^-}\frac{d^p}{dt^p}\boldsymbol{u}(t) = \lim_{t\to t_1^+}\frac{d^p}{dt^p}\boldsymbol{u}(t)$, for $p = 1, 2, \dots$. ∎

Remark 5. The consensus scheme (26) for second-order dynamic agents has been developed based on a backstepping-like technique.[39] In particular, we first cancel out the first-order derivative of the reference control signal $\boldsymbol{\phi}_1(t)$ in $\dot{\boldsymbol{z}}(t)$ in (24), and then add a FWAT control term, that is, $-\frac{\eta_2}{t_1-t}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}})$, to the control $\boldsymbol{u}$ so that $\boldsymbol{z}\to\mathbf{0}$ as $t\to t_1$. After that, the system is reduced to the first-order counterpart, and hence an arbitrary preset time consensus is subsequently achieved. Such a consensus control scheme is not straightforwardly applicable for those fixed-time controllers that use the signum function, $\mathrm{sgn}(\boldsymbol{s})$, or negative power feedback terms, for example, $\frac{\boldsymbol{s}}{\|\boldsymbol{s}\|^\alpha}$, for some sliding surface vector $\boldsymbol{s}$ and positive constant α, because the derivative of the associated reference controls contain a singularity at $\boldsymbol{s} = \mathbf{0}$. A consensus law with a complex switching sliding mode was investigated in Reference 24 to avoid such a singularity.

Remark 6. Note that after the pre-specified time t_f, within which the consensus has been achieved by the agents, the proposed consensus protocols (5) and (26) provide zero control efforts. Due to this, they may be more suitable for stabilization control problems, rather than tracking control ones that require the agents to track time-varying reference signals. For the former, to stabilize the system to multiple equilibrium points successively, one can employ the proposed FWAT control protocols for successive prespecified time intervals, namely $[t_0, t_{f1}), [t_{f1}, t_{f2}), [t_{f2}, t_{f3}), \dots,$ within each time interval a certain desired equilibrium point is stabilized.

4.2 | Boundedness of the system state

Let us consider the following perturbed system

$$\dot{\boldsymbol{x}} = -\boldsymbol{\phi}_1(t, \boldsymbol{x}) + \boldsymbol{z}(t), \ \ t\in[t_0, t_1] \tag{30}$$

with the auxiliary vector $\boldsymbol{z}(t)$, which is defined in (23), being a perturbed signal. The perturbed input $\boldsymbol{z}(t)$ converges to zero in a free-will arbitrary prespecified time t_1 (Theorem 2). Thus, $\|\boldsymbol{z}(t)\|$ is also *absolutely integrable* as the area under the curve $\|\boldsymbol{z}(t)\|$ between $t\in[t_0, t_f]$ is finite, that is, $\int_{t_0}^t \|\boldsymbol{z}(\tau)\| d\tau < \infty, \forall t \geq 0$.

Let $\boldsymbol{P} = (\boldsymbol{I}_n - \mathbf{1}_n\mathbf{1}_n^\top/n)$ be the orthogonal projection onto $\ker(\mathbf{1}_n)$. Note that one can write $\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x} + (\boldsymbol{I}_n - \boldsymbol{P})\boldsymbol{x}$. Thus, we bound these two components of $\boldsymbol{x}$ in what follows.

By left-multiplying by $\boldsymbol{P}$ on both sides of (30) and letting $\boldsymbol{x}^{\parallel} := \boldsymbol{P}\boldsymbol{x}$, we have

$$\begin{aligned}\dot{\boldsymbol{x}}^{\parallel} &= \frac{\eta}{t_f - t}\boldsymbol{P}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}} + \boldsymbol{P}\boldsymbol{z}(t)\\ &= \frac{\eta}{t_f - t}\mathcal{L}\mathrm{e}^{-\mathcal{L}\boldsymbol{x}^{\parallel}} + \boldsymbol{P}\boldsymbol{z}(t),\end{aligned} \tag{31}$$

where we have used the relations $\boldsymbol{P}\mathcal{L} = \mathcal{L}\boldsymbol{P} = \mathcal{L}$ and $\mathcal{L}\boldsymbol{x} = \mathcal{L}\boldsymbol{P}\boldsymbol{x}$. Note importantly that $\mathbf{1}_n^\top\dot{\boldsymbol{x}}^{\parallel} = 0$ for all time t. Thus, we obtain the following lemma whose proof is given in Appendix A.

Lemma 6. *The average point* $\overline{\boldsymbol{x}}^{\parallel} := \left(\mathbf{1}_n^\top\boldsymbol{x}^{\parallel}(t_0)/n\right)\mathbf{1}_n$ *of the nominal system* $\dot{\boldsymbol{x}}^{\parallel} = -\boldsymbol{\phi}_1(t, \boldsymbol{x}^{\parallel})$ *of (31) is free-will arbitrary time stable, and the perturbed system (31) is input to state stable with respect to the vanishing input* $\boldsymbol{P}\boldsymbol{z}(t)$.

Let $\boldsymbol{x}^{\perp} := (\boldsymbol{I}_n - \boldsymbol{P})\boldsymbol{x}$. Left-multiplying by $(\boldsymbol{I}_n - \boldsymbol{P})$ on both sides of (30) yields

$$\dot{\boldsymbol{x}}^{\perp} = (\boldsymbol{I}_n - \boldsymbol{P})\boldsymbol{z}, \tag{32}$$

where we have used the relation $(\boldsymbol{I}_n - \boldsymbol{P})\mathcal{L} = \mathbf{1}_n\mathbf{1}_n^\top\mathcal{L}/n = \mathbf{0}$. Then, it follows from the preceding equation that

$$\int_{t_0}^{t} d\boldsymbol{x}^{\perp} = (\boldsymbol{I}_n - \boldsymbol{P})\int_{t_0}^{t} \boldsymbol{z}(\tau)d\tau$$
$$\Leftrightarrow \boldsymbol{x}^{\perp}(t) - \boldsymbol{x}^{\perp}(t_0) = (\boldsymbol{I}_n - \boldsymbol{P})\int_{t_0}^{t} \boldsymbol{z}(\tau)d\tau$$
$$\Leftrightarrow ||\boldsymbol{x}^{\perp}(t) - \boldsymbol{x}^{\perp}(t_0)|| \leq \int_{t_0}^{t} ||\boldsymbol{z}(\tau)||d\tau < \infty.$$

It follows that $\boldsymbol{x}^{\perp}(t)$ is bounded for all time $t \in [t_0, t_1]$. Thus, the following result is obtained directly from the above analysis.

Lemma 7. *Consider the system of double-integrator agents (22) with a connected communication graph $\mathcal{G}$. Under the consensus law (26) with $\eta > 1/\lambda_2^2$ and $\eta_2 > 1$, the state vector $\boldsymbol{x}(t)$ is bounded for all time $t \in [t_0, t_1]$.*

4.3 | Extension to FWAT consensus of p-order integrator agents

The above-mentioned control design method in Remark 5 can be used to design FWAT consensus of agents with higher-order dynamics. To proceed, we consider the pth order chain of integrators ($p \geq 2$)

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2,\ \dot{\boldsymbol{x}}_2 = \boldsymbol{x}_3,\ \dots,\ \dot{\boldsymbol{x}}_p = \boldsymbol{u}_p, \tag{33}$$

where $\boldsymbol{u}_p \in \mathbb{R}^n$ is the control input, and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_p \in \mathbb{R}^n$ denote the state vectors of the multi-agent system. The objective is to achieve a FWAT consensus in the entries of the first state vector $\boldsymbol{x}_1 := [x_{1,1}, \dots, x_{1,n}]^{\top} \in \mathbb{R}^n$, or that is, $x_{1,1} = x_{1,2} = \cdots = x_{1,n}$, where each entry $x_{1,i}$ corresponds to an agent i.

To proceed, we first introduce an auxiliary variable $\boldsymbol{z}_p = \boldsymbol{x}_p - \boldsymbol{u}_{p-1}$, where $\boldsymbol{u}_{p-1} \in \boldsymbol{R}^n$ is an additional (smooth) control vector to be designed. Define the function $\boldsymbol{\psi} : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$\boldsymbol{\psi}(t_p, \boldsymbol{z}_p) := -\frac{\eta}{t_p - t}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}_p}),$$

where $\eta > 1$ and $t_p > 0$ is a pre-chosen time. Then, following a similar design method used for the case of double-integrator agents (26), we design the control $\boldsymbol{u}_p$ as

$$\boldsymbol{u}_p = \begin{cases} \dot{\boldsymbol{u}}_{p-1} + \boldsymbol{\psi}(t_p, \boldsymbol{z}_p), & \text{if } 0 \leq t < t_p, \\ \dot{\boldsymbol{u}}_{p-1}, & \text{if } t \geq t_p. \end{cases} \tag{34}$$

It can be shown that $\boldsymbol{z}_p \to \mathbf{0}$ at $t = t_p$ and stays thereafter. As a result, we obtain the reduced $(p-1)$th order chain of integrators

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2,\ \dots,\ \dot{\boldsymbol{x}}_{p-1} = \boldsymbol{x}_p = \boldsymbol{u}_{p-1},\ \forall t \geq t_p.$$

We can now repeatedly apply the control design procedure for the reduced systems in a successive way. In particular, we define the auxiliary vectors, $\boldsymbol{z}_j = \boldsymbol{x}_j - \boldsymbol{u}_{j-1}$, and design the controls

$$\boldsymbol{u}_j = \begin{cases} \dot{\boldsymbol{u}}_{j-1} + \boldsymbol{\psi}(t_j, \boldsymbol{z}_j), & \text{if } t_{j+1} \leq t < t_j, \\ \boldsymbol{u}_j = \dot{\boldsymbol{u}}_{j-1}, & \text{if } t \geq t_j, \end{cases} \tag{35}$$

for $j = p-1, \dots, 3$, with the prespecified times satisfying $t_3 > t_4 > \cdots > t_p$. Thus, it can be shown that when $t \geq t_3$, the pth order chain of integrators (33) is reduced to the second-order system $\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2,\ \dot{\boldsymbol{x}}_2 = \boldsymbol{u}_2$. Therefore, the control input $\boldsymbol{u}_2$ can be designed the same as one in (26) and a FWAT consensus can be achieved.

4.4 | Robust FWAT consensus of double-integrator agents under disturbances

We now consider the second-order dynamics of the n agents in the presence of the disturbance as follows

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \ \dot{\boldsymbol{v}}(t) = \boldsymbol{u}(t) + \boldsymbol{d}(t), \tag{36}$$

where $\boldsymbol{d} \in \mathbb{R}^n$ is the matched disturbance. We will also use the assumption that the disturbance is bounded, that is, there exists a positive constant $\overline{d} > 0$ satisfying $\|\boldsymbol{d}\|_\infty \le \overline{d}$. Consider the consensus law

$$\boldsymbol{u} = \boldsymbol{u}^{nom} - \beta \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}, \tag{37}$$

where the nominal control $\boldsymbol{u}^{nom}$ is given the same as the consensus control in (26), the tracking error vector $\boldsymbol{z}$ is defined in (23), and the constant gain β is assumed to satisfy $\beta > \sqrt{n}\|\boldsymbol{d}\|_\infty = \sqrt{n\overline{d}}$. It then follows from (24), (36), and (37) that the derivative of the Lyapunov function $V = \boldsymbol{z}^\top \boldsymbol{z}$ is given as

$$\begin{aligned}
\dot{V} &= 2\boldsymbol{z}^\top \dot{\boldsymbol{z}} \\
&= 2\boldsymbol{z}^\top \left(\dot{\boldsymbol{v}} + \frac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{x}} \boldsymbol{v} + \frac{\partial \boldsymbol{\phi}_1}{\partial t} \right) \\
&= -2\frac{\eta_2}{t_1 - t} \boldsymbol{z}^\top (\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) + 2\boldsymbol{z}^\top \left(-\beta \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|} + \boldsymbol{d} \right) \\
&\le -2\frac{\eta_2}{t_1 - t} \boldsymbol{z}^\top (\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) - 2\|\boldsymbol{z}\| \left(\beta - \|\boldsymbol{d}\|\right) \\
&\le -2\frac{\eta_2}{t_1 - t} \boldsymbol{z}^\top (\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}) - 2\|\boldsymbol{z}\| \left(\beta - \sqrt{n}\|\boldsymbol{d}\|_\infty\right) \\
&\le -2\frac{\eta_2}{t_1 - t} \boldsymbol{z}^\top (\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}),
\end{aligned} \tag{38}$$

where in the first and second inequalities we have used Holder's inequality and the norm inequality $\|\boldsymbol{d}\| \le \sqrt{n}\|\boldsymbol{d}\|_\infty$, respectively. By following similar lines as in Proof of Theorem 2, one can show that $\boldsymbol{z} \to \mathbf{0}$ within the chosen time t_1, and the agents's states converge to a consensus within the prespecified time t_f if $\eta > 1/\lambda_2^2$.

Example 3:

An example of FWAT consensus of four agents under (26) with $\eta = \eta_2 = 2, t_1 = 3$ s and $t_f = 4$ s is given in Figure 3. The communication graph of the agents is a ring graph. In the simulation, the states of the agents $x_i(0), i = 1, 2, 3, 4$, are initialized randomly in $[0, 1]$ and $v_i(0)$, for $i = 1, 2, 3, 4$, are chosen randomly in $[0, 0.5]$. It is observed from Figure 3 that the tracking vector $\boldsymbol{z} = \boldsymbol{v} + \boldsymbol{\phi}_1$ converges to zero at $t_1 = 3$ s and the agent states achieve a consensus within the prespecified time $t_f = 4$ s.

For comparison, we also carry out a simulation for FWAT consensus of double-integrator agents under (37) in the presence of disturbances $d_i = 0.1\sin(t), i = 1, 2, 3, 4$. The control again is selected as $\beta = 0.25$ so that it satisfies $\beta > \sqrt{n\overline{d}} = 0.2$. The system graph, the agents' initial velocities and positions, and other controller's design parameters are the same as before. Simulation results are provided in Figure 4. It can be seen that the tracking vector $\boldsymbol{z} = \boldsymbol{v} + \boldsymbol{\phi}_1$ converges to zero within $t_1 = 3$ s and the agent states achieve a consensus within the prespecified time $t_f = 4$ s. Once the tracking error vector $\boldsymbol{z}$ has converged to zero (within $t_1 = 3$ s), the control inputs of the agents experience chattering due to the discontinuous negative power feedback term $\frac{\boldsymbol{z}}{\|\boldsymbol{z}\|}$ in (37), as shown in Figure 4D.

5 | APPLICATION TO FWAT FORMATION CONTROL OF MOBILE ROBOTS

In this section, we present an application of the proposed free-will arbitrary time consensus scheme (26) to the FWAT formation control of multiple nonholonomic agents in the two-dimensional plane. In particular, by using feedback linearization, the kinematic model of the agents is first transformed into the second-order integrator dynamics. Then, the

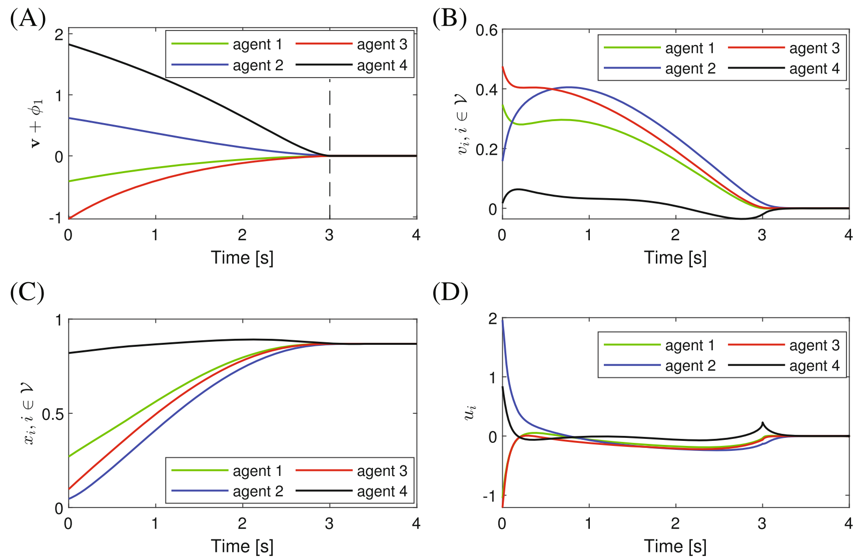

FIGURE 3 Consensus of four double-integrator agents under the control law (26) in the absence of disturbances with $\eta = 2.6, \eta_2 = 1.6, t_1 = 3$ s, and $t_f = 4$ s. (A) Tracking error vector $\mathbf{z}(t)$ versus time. (B) Velocities of the agents versus time. (C) The agents' states $x_i(t)$ versus time. (D) Controls $u_i(t), i = 1, 2, 3, 4$, versus time

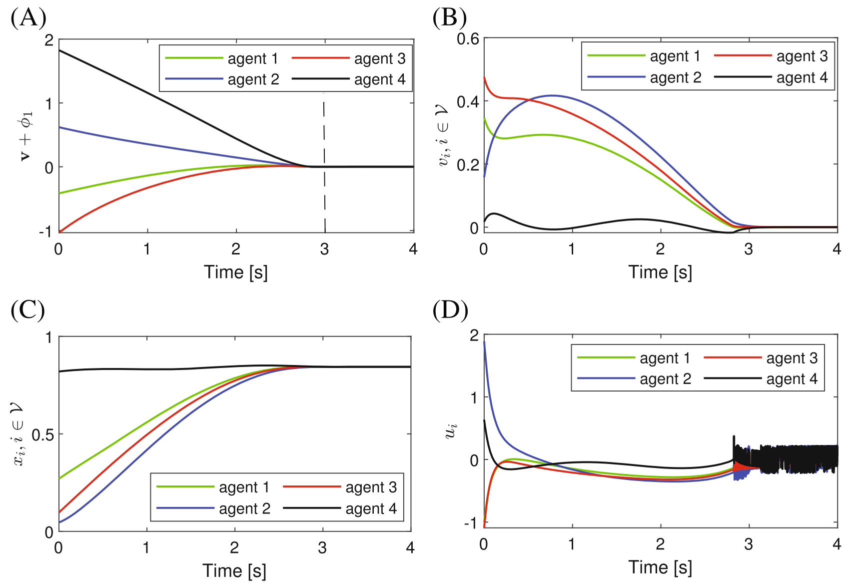

FIGURE 4 Robust prespecified time consensus of four double-integrator agents under the control law (37) in the presence of bounded disturbances with $\eta = 2.6, \eta_2 = 1.6, t_1 = 3$ s, and $t_f = 4$ s. (A) Tracking error vector $\mathbf{z}(t)$ versus time. (B) Velocities of the agents versus time. (C) The agents' states $x_i(t)$ versus time. (D) Controls $u_i(t), i = 1, 2, 3, 4$, versus time

results in Section 4 are used to design distributed control laws to stabilize the desired geometric pattern of the agents' hand positions in free-will arbitrary time.

5.1 | Two-wheeled mobile robots

The motion of each mobile robot at the kinematic level is given as (see Figure 5)

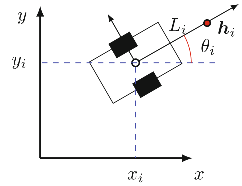

FIGURE 5 A mobile robot in $\mathbb{R}^2$

$$\dot{\boldsymbol{p}}_i = \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} v_i, \ \dot{\theta}_i = \omega_i, \tag{39}$$

where $\boldsymbol{p}_i = [x_i, y_i]^\top$ denotes the coordinates of the robot i's center location, θ_i is the robot i's heading angle, and v_i and ω_i are respectively the linear and angular velocity of the robot.

The hand position (or tool position) $\boldsymbol{h}_i \in \mathbb{R}^2$ (see Figure 5) is given as

$$\boldsymbol{h}_i = \begin{bmatrix} h_{ix} \\ h_{iy} \end{bmatrix} = \boldsymbol{p}_i + \begin{bmatrix} \cos(\theta_i) \\ \sin(\theta_i) \end{bmatrix} L_i, \tag{40}$$

where L_i is the distance from the hand location to the robot i's center point. The second derivative of $\boldsymbol{h}_i$ can be obtained as

$$\ddot{\boldsymbol{h}}_i = \begin{bmatrix} \cos(\theta_i) & -L_i \sin(\theta_i) \\ \sin(\theta_i) & L_i \cos(\theta_i) \end{bmatrix} \begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} + \begin{bmatrix} -\sin(\theta_i) v_i \omega_i - L_i \cos(\theta_i) \omega_i^2 \\ \cos(\theta_i) v_i \omega_i - L_i \sin(\theta_i) \omega_i^2 \end{bmatrix}. \tag{41}$$

By using the following change of variable[9] and feedback linearization

$$\begin{bmatrix} \dot{v}_i \\ \dot{\omega}_i \end{bmatrix} = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\frac{1}{L_i} \sin(\theta_i) & \frac{1}{L_i} \cos(\theta_i) \end{bmatrix} \left(\boldsymbol{u}_i - \begin{bmatrix} -\sin(\theta_i) v_i \omega_i - L_i \cos(\theta_i) \omega_i^2 \\ \cos(\theta_i) v_i \omega_i - L_i \sin(\theta_i) \omega_i^2 \end{bmatrix} \right), \tag{42}$$

where $\boldsymbol{u}_i \in \mathbb{R}^2$ is the control input of agent i, which will be designed later, we obtain

$$\ddot{\boldsymbol{h}}_i = \boldsymbol{u}_i, \tag{43}$$

which is in the form of (22).

5.2 | Formation control protocol

Consider a system of four mobile robots in $\mathbb{R}^2$ whose local interaction is described by a ring graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the agent index set $\mathcal{V} = \{1, 2, 3, 4\}$ and the edge set $\mathcal{E} = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. The system aims to form a square of side length of 1m, which is specified by the set of desired displacements of the robots' relative hand positions $\{\boldsymbol{h}_{12}^* = \boldsymbol{h}_{42}^* = [1, 0]^\top, \boldsymbol{h}_{41}^* = \boldsymbol{h}_{32}^* = [0, 1]^\top\}$, where $\boldsymbol{h}_{ij}^* = \boldsymbol{h}_j^* - \boldsymbol{h}_i^*$. The robots start at rest and from locations chosen in $[0, 3] \times [0, 3]$ (m). The initial heading angles of the agents are selected as $\theta_1 = 0, \theta_2 = \pi/2, \theta_3 = \pi/3$, and $\theta_4 = \pi/6$ (rad), respectively.

Define

$$\boldsymbol{u} = [\boldsymbol{u}_1^\top, \boldsymbol{u}_2^\top, \boldsymbol{u}_3^\top, \boldsymbol{u}_4^\top]^\top \in \mathbb{R}^8, \tag{44}$$

$$\boldsymbol{h} = [\boldsymbol{h}_1^\top, \boldsymbol{h}_2^\top, \boldsymbol{h}_3^\top, \boldsymbol{h}_4^\top]^\top \in \mathbb{R}^8, \tag{45}$$

$$\overline{\mathcal{L}} = \mathcal{L} \otimes \boldsymbol{I}_2, \ \boldsymbol{\phi}_1 = -\frac{\eta}{t_f - t} \overline{\mathcal{L}} e^{-\overline{\mathcal{L}}(\boldsymbol{h} - \boldsymbol{h}^*)}, \tag{46}$$

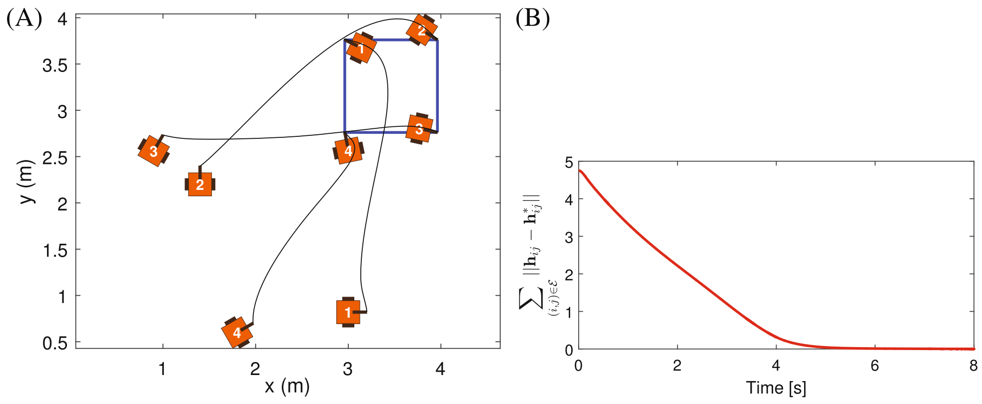

FIGURE 6 Formation control of four mobile robots under (48) with $\eta = \eta_2 = 2$, $t_1 = 4$ s, and $t_f = 8$ s. Trajectories of the robots (A). Total displacement error versus time (B)

and $\boldsymbol{z} = \dot{\boldsymbol{h}} + \boldsymbol{\phi}_1$ with

$$\dot{\boldsymbol{h}}_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)L_i \\ \sin(\theta_i) & \cos(\theta_i)L_i \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}. \tag{47}$$

Then, we design the control input as

$$\boldsymbol{u} = \begin{cases} -\dfrac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{h}}\dot{\boldsymbol{h}} - \dfrac{\partial \boldsymbol{\phi}_1}{\partial t} - \dfrac{\eta_2}{t_1 - t}(\mathbf{1}_n - \mathrm{e}^{-\boldsymbol{z}}), & \text{if } t_0 \le t < t_1 \\ -\dfrac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{h}}\dot{\boldsymbol{h}} - \dfrac{\partial \boldsymbol{\phi}_1}{\partial t}, & \text{if } t_1 \le t < t_f \\ \mathbf{0}, & \text{otherwise}, \end{cases} \tag{48}$$

where the partial derivative terms are respectively given as

$$\frac{\partial \boldsymbol{\phi}_1}{\partial \boldsymbol{h}} = \frac{\eta}{t_f - t}\overline{\mathcal{L}}\mathrm{diag}\left(\mathrm{e}^{-\overline{\mathcal{L}}\boldsymbol{x}}\right)\overline{\mathcal{L}}, \text{ and } \frac{\partial \boldsymbol{\phi}_1}{\partial t} = -\frac{\eta}{(t_f - t)^2}\overline{\mathcal{L}}\mathrm{e}^{-\overline{\mathcal{L}}\boldsymbol{x}}.$$

Simulation results of formation control of four mobile robots under the control law (48) with $\eta = \eta_2 = 2$, $t_1 = 4$ s, and $t_f = 8$ s are provided in Figure 6. A video of the simulation can be found in https://youtu.be/rVPExz7qbGk. It can be seen that the robots' hand positions form a square (in blue) within the prespecified time $t_f = 8$ s.

6 | CONCLUSION

In this work, free-will arbitrary time consensus schemes were presented for MASs of both single- and double- (or high) order integrator agents, possibly with switching graph topologies, and in the presence of matched disturbances. The average consensus protocol for systems of single-integrator agents was introduced to remedy the technical issues associated with the consensus protocol in Reference 28. All the proposed consensus schemes possess distributed nature which is favored in problems related to MASs where only local communication and sensing between neighboring agents are employed. Further, the bound of the convergence time of the proposed consensus protocols is explicitly available and can be chosen arbitrarily independent of the initial condition or any other parameters. Finally, an application of the proposed consensus scheme in FWAT formation control of mobile agents was presented and simulation results were also provided to validate the theoretical development.

For future works, it will be interesting to investigate FWAT coordination control tasks such as attitude synchronization and formation tracking control, based on the results developed in this work, for agents with more practical models.

Further, the discretized implementation of the proposed protocols based on the implicit Euler discretization method[42-44] is worthy of investigation. Another possible research direction is to address the FWAT consensus of MASs with general directed graphs containing a spanning tree.

ACKNOWLEDGMENT

The work of Hyo-Sung Ahn is supported by the National Research Foundation (NRF) of Korea under the Grant NRF-2022R1A2B5B03001459. The work of Quoc Van Tran was supported by the BK21 FOUR Program of the National Research Foundation Korea (NRF) Grant funded by the Ministry of Education (MOE). The work of Nam Hoai Nguyen was supported by the Ministry of Science and Technology in Vietnam under the Grant KC-4.0-34/1925.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

ENDNOTE

*Since $(B + \mathcal{L}_f)$ is positive definite no left-multiplication, for example, by $\mathcal{L}$ as in (9), will be needed here.

ORCID

Quoc Van Tran https://orcid.org/0000-0002-7962-0338
Minh Hoang Trinh https://orcid.org/0000-0001-5736-6693
Nam Hoai Nguyen https://orcid.org/0000-0002-0391-3775
Hyo-Sung Ahn https://orcid.org/0000-0002-7939-0093

REFERENCES

1. Ren W, Atkins E. Distributed multi-vehicle coordinated control via local information exchange. *Int J Robust Nonlinear Control*. 2007;17:1002-1033.
2. Olfati-Saber R, Murray RM. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans Automat Contr*. 2004;49(9):1520-1533.
3. Mesbahi M, Egerstedt M. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press; 2010.
4. Van Tran Q, Sun Z, Anderson BD, Ahn HS. Distributed computation of graph matching in multi-agent networks. Proceedings of the 59th IEEE Conference on Decision Control (CDC); 2020:3139-3144.
5. Tran QV, Trinh MH, Zelazo D, Mukherjee D, Ahn H-S. Finite-time bearing-only formation control via distributed global orientation estimation. *IEEE Trans Control Netw Syst*. 2019;2(6):702-712.
6. John Y, Oh K-K, Fidan B, Ahn H-S. Fixed-time orientation estimation and network localisation of multi-agent systems. *IET Control Theory Appl*. 2021;15(1):64-76.
7. Bohn J, Sanyal AK. Almost global finite-time stabilization of rigid body attitude dynamics using rotation matrices. *Int J Robust Nonlinear Control*. 2016;26(9):2008-2022. doi:10.1002/rnc.3399
8. Xu C, Wu B, Wang D, Zhang Y. Distributed fixed-time output-feedback attitude consensus control for multiple spacecraft. *IEEE Trans Aerosp Electron Syst*. 2020;56(6):4779-4795. doi:10.1109/TAES.2020.3003119
9. Khoo S, Xie L, Man Z. Robust finite-time consensus tracking algorithm for multirobot systems. *IEEE/ASME Trans Mechatron*. 2009;14(2):219-228. doi:10.1109/TMECH.2009.2014057
10. Ou M, Du H, Li S. Finite-time formation control of multiple nonholonomic mobile robots. *Int J Robust Nonlinear Control*. 2014;24(1):140-165. doi:10.1002/rnc.2880
11. Chen G, Ren J, Feng EN. Distributed finite-time economic dispatch of a network of energy resources. *IEEE Trans Smart Grid*. 2017;8(2):822-832. doi:10.1109/TSG.2016.2516017
12. Wang Y, Song Y, Hill DJ, Krstic M. Prescribed-time consensus and containment control of networked multiagent systems. *IEEE Trans Cybern*. 2019;49(4):1138-1147. doi:10.1109/TCYB.2017.2788874
13. Mehdifar F, Hashemzadeh F, Baradarannia M, Queiroz M. Finite-time rigidity-based formation maneuvering of multiagent systems using distributed finite-time velocity estimators. *IEEE Trans Cybern*. 2019;49(12):4473-4484. doi:10.1109/TCYB.2018.2876608
14. Wang L, Xiao F. Finite-time consensus problems for networks of dynamic agents. *IEEE Trans Automat Contr*. 2010;55(4):950-955. doi:10.1109/TAC.2010.2041610
15. Shi X, Cao J, Wen G, Huang W. Finite-time consensus of opinion dynamics and its applications to distributed optimization over digraph. *IEEE Trans Cybern*. 2019;49(10):3767-3779.
16. Bhat S, Bernstein D. Finite-time stability of continuous autonomous systems. *SIAM J Control Optim*. 2000;38(3):751-766.

17. Zou W, Shi P, Xiang Z, Shi Y. Finite-time consensus of second-order switched nonlinear multi-agent systems. *IEEE Trans Neural Netw Learn Syst*. 2020;31(5):1757-1762. doi:10.1109/TNNLS.2019.2920880
18. Chen D, Lu T, Liu X, Yu W. Finite-time consensus of multiagent systems with input saturation and disturbance. *Int J Robust and Nonlinear Control*. 2021;31(6):2097-2109. doi:10.1002/rnc.5029
19. Wang H, Yu W, Ren W, Lü J. Distributed adaptive finite-time consensus for second-order multiagent systems with mismatched disturbances under directed networks. *IEEE Trans Cybern*. 2021;51(3):1347-1358. doi:10.1109/TCYB.2019.2903218
20. Polyakov A. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Trans Automat Contr*. 2012;57(8):2106-2110.
21. Wang H, Yu W, Wen G, Chen G. Fixed-time consensus of nonlinear multi-agent systems with general directed topologies. *IEEE Trans Circuits Syst II Exp Briefs*. 2019;66(9):1587-1591. doi:10.1109/TCSII.2018.2886298
22. Zou W, Qian K, Xiang Z. Fixed-time consensus for a class of heterogeneous nonlinear multiagent systems. *Trans Circuits Syst II Exp Briefs*. 2020;67(7):1279-1283. doi:10.1109/TCSII.2019.2930648
23. Dávila J, Pisano A. On the fixed-time consensus problem for nonlinear uncertain multiagent systems under switching topology. *Int J Robust Nonlinear Control*. 2021;31(9):3841-3858. doi:10.1002/rnc.5130
24. Li D, Yu H, Tee KP, Wu Y, Ge SS, Lee TH. On time-synchronized stability and control. *IEEE Trans Syst Man Cybern Syst*. 2021;52(4):1-14. doi:10.1109/TSMC.2021.3050183
25. Trinh MH, Truong NT, Nguyen NH, Ahn HS. Fixed-time network localization based on bearing measurements. Proceedings of the 2020 American Control Conference (ACC); 2020:903-908.
26. Li Z, Tnunay H, Zhao S, Meng W, Xie SQ, Ding Z. Bearing-only formation control with prespecified convergence time. *IEEE Trans Cybern*. 2020;1-10:620-629. doi:10.1109/TCYB.2020.2980963
27. Pal AK, Kamal S, Nagar SK, Bandyopadhyay B, Fridman L. Design of controllers with arbitrary convergence time. *Automatica*. 2020;112:108710.
28. Pal AK, Kamal S, Yu X, Nagar SK, Xiong X. Free-will arbitrary time consensus for multiagent systems. *IEEE Trans Cybern*. 2020;1–11:4636-4646. doi:10.1109/TCYB.2020.3032217
29. Nguyen TT, Trinh MH, Nguyen CV, Nguyen NH, Dang MV. Coordination of multi-agent systems with arbitrary convergence time. *IET Control Theory Appl*. 2021;15:900-909.
30. Yu S, Long X. Finite-time consensus for second-order multi-agent systems with disturbances by integral sliding mode. *Automatica*. 2015;54:158-165. doi:10.1016/j.automatica.2015.02.001
31. Yu W, Wang H, Cheng F, Yu X, Wen G. Second-order consensus in multiagent systems via distributed sliding mode control. *IEEE Trans Cybern*. 2017;47(8):1872-1881. doi:10.1109/TCYB.2016.2623901
32. Feng Y, Yu X, Han F. On nonsingular terminal sliding-mode control of nonlinear systems. *Automatica*. 2013;49(6):1715-1722. doi:10.1016/j.automatica.2013.01.051
33. Zuo Z. Nonsingular fixed-time consensus tracking for second-order multi-agent networks. *Automatica*. 2015;54:305-309. doi:10.1016/j.automatica.2015.01.021
34. Zuo Z, Tian B, Defoort M, Ding Z. Fixed-time consensus tracking for multiagent systems with high-order integrator dynamics. *IEEE Trans Automat Contr*. 2018;63(2):563-570. doi:10.1109/TAC.2017.2729502
35. Utkin V., Shi J.. Integral sliding mode in systems operating under uncertainty conditions. Proceedings of the 35th IEEE Confer. Decision Control (CDC); 1996:4591-4596.
36. Bullo F. Lectures on network systems. Kindle direct publishing; 1.4 ed.2020. With contributions by J. Cortes, F. Dorfler, and S. Martinez
37. Ahn H-S. *Formation Control: Approaches for Distributed Agents*. Springer International Publishing; 2019.
38. Tran QV, Ahn H-S. Distributed formation control of mobile agents via global orientation estimation. *IEEE Trans Control Netw Syst*. 2020;4(7):1654-1664.
39. Khalil HK. *Nonlinear Systems*. 3rd ed. Prentice Hall; 2002.
40. Ji H, Zhang H-T, Ye Z, Zhang H, Xu B, Chen G. Stochastic consensus control of second-order nonlinear multiagent systems with external disturbances. *IEEE Trans Control Netw Syst*. 2018;5(4):1585-1596. doi:10.1109/TCNS.2017.2736959
41. Filippov AF. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publishers; 1988.
42. Polyakov A, Efimov D, Brogliato B. Consistent discretization of finite-time and fixed-time stable systems. *SIAM J Control Optim*. 2019;57(1):78-103. doi:10.1137/18M1197345
43. Xiong X, Sachan A, Huang R, Kumar PA, Kamal S, Chen W. Implicit-Euler based digital implementation for constrained stabilization of second-order systems. *Inter J Robust Nonlinear Control*. 2021;31(11):5086-5100. doi:10.1002/rnc.5525
44. Xiong X, Pal AK, Liu Z, Kamal S, Huang R, Lou Y. Discrete-time adaptive super-twisting observer with predefined arbitrary convergence time. *IEEE Trans Circuits Syst II Exp Briefs*. 2021;68(6):2057-2061. doi:10.1109/TCSII.2020.3042633

How to cite this article: Tran QV, Trinh MH, Nguyen NH, Ahn H-S. Free-will arbitrary time consensus protocols with diffusive coupling. *Int J Robust Nonlinear Control*. 2022;32(15):8711-8731. doi: 10.1002/rnc.6267

APPENDIX A. PROOF OF LEMMA 6

The free-will arbitrary time convergence of $\bar{\boldsymbol{x}}^{\parallel}$ of the nominal system $\dot{\boldsymbol{x}}^{\parallel} = -\boldsymbol{\phi}_1\left(t, \boldsymbol{x}^{\parallel}\right)$ follows from a similar argument as in Proof of Theorem 1.

Let $\boldsymbol{\delta} = \boldsymbol{x}^{\parallel} - \bar{\boldsymbol{x}}^{\parallel}$; it follows that $\mathbf{1}_n^{\top}\boldsymbol{\delta} = 0$ along the trajectory of (31). Thus, the derivative of the Lyapunov function $V = \delta^{\top}\delta$ is given as

$$\begin{aligned}\dot{V} &= -\frac{2\eta}{t_f - t}\boldsymbol{\delta}^{\top}\mathcal{L}\left(\mathbf{1}_n - \mathrm{e}^{-\mathcal{L}\boldsymbol{\delta}}\right) + 2\boldsymbol{\delta}^{\top}\boldsymbol{P}\boldsymbol{z}(t)\\ &\leq -\frac{2\eta}{t_f - t}||\mathcal{L}\boldsymbol{\delta}||\left(\mathbf{1}_n - \mathrm{e}^{-||\mathcal{L}\boldsymbol{\delta}||}\right) + 2\|\boldsymbol{\delta}\|\|\boldsymbol{z}(t)\|,\end{aligned} \tag{A1}$$

where the inequality follows from (3) and the inequality $||\boldsymbol{P}\boldsymbol{z}(t)|| \leq ||\boldsymbol{z}(t)||$. Since $\mathbf{1}_n^{\top}\boldsymbol{\delta} = 0$ we have $\lambda_2(\mathcal{L})\sqrt{V} \leq ||\mathcal{L}\boldsymbol{\delta}||$. As a result, it follows from (A1) that

$$\dot{V} \leq -\frac{2\eta}{t_f - t}\lambda_2\sqrt{V}\left(1 - \mathrm{e}^{-\lambda_2\sqrt{V}}\right) + 2\sqrt{V}||\boldsymbol{z}(t)||.$$

Now, let $\xi = \lambda_2\sqrt{V}$; then it follows from the preceding inequality we have that

$$\begin{aligned}\dot{\xi} &= \lambda_2\frac{\dot{V}}{2\sqrt{V}}\\ &\leq -\frac{\eta\lambda_2^2}{t_f - t}\left(1 - \mathrm{e}^{-\xi}\right) + \lambda_2||\boldsymbol{z}(t)||.\\ &\leq \lambda_2||\boldsymbol{z}(t)||.\end{aligned}$$

From the comparison lemma,[39] we have

$$\xi(t) \leq \int_{t_0}^{t}||\boldsymbol{z}(\tau)||d\tau + \xi(0) < \infty,$$

for all $t \in [t_0, t_1]$. This shows that V is bounded and so is $\boldsymbol{x}^{\parallel}(t)$ for all $t \in [t_0, t_1]$. Thus, the perturbed system (31) is input to state stable with respect to the vanishing input $\boldsymbol{z}(t)$.